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Muonium Decay

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Abstract

Modifications of the μ^+ lifetime in matter due to muonium ($M = \mu^+e^-$) formation and other medium effects are examined. Muonium and free μ^+ decay spectra are found to differ at $\mathcal{O}(\alpha m_e/m_\mu)$ from Doppler broadening and $\mathcal{O}(\alpha^2 m_e/m_\mu)$ from the Coulomb bound state potential. However, both types of corrections are shown to cancel in the total decay rate due to Lorentz and gauge invariance respectively, leaving a very small time dilation lifetime difference, $(\tau_M - \tau_{\mu^+})/\tau_{\mu^+} = \frac{\alpha^2 m_e^2}{2m_\mu^2} \simeq 6\times 10^{-10}$, as the dominant bound state effect. It is argued that other medium effects on the stopped μ^+ lifetime are similarly suppressed.

The muon lifetime, τ_μ, is very well measured. Its current world average [1]

$$\tau_\mu = 2.197035(40)\times 10^{-6}\ \mathrm{sec} \tag{1}$$

exhibits an uncertainty of only 18 ppm. From that lifetime, the Fermi constant (denoted here by G_μ) is determined via the defining relationship [2]

$$\begin{aligned}\tau_\mu^{-1} &= \Gamma(\mu\to\mathrm{all}) = \frac{G_\mu^2 m_\mu^5}{192\pi^3} f\left(\frac{m_e^2}{m_\mu^2}\right)\left(1+\frac{3}{5}\frac{m_\mu^2}{m_W^2}\right)(1+\mathrm{R.C.}),\\ f(x) &\equiv 1-8x+8x^3-x^4-12x^2\ln x,\end{aligned} \tag{2}$$

where m_μ is the muon pole mass and R.C. stands for QED radiative corrections to muon decay as calculated in an effective local V-A theory. Other standard model electroweak loop corrections as well as possible "new physics" effects are absorbed in G_μ.

The R.C. in (2) have been computed [2,3] through $\mathcal{O}(\alpha^2)$ and higher order logs have been obtained using the renormalization group [4]. Altogether, one finds [5,6]

$$
\begin{aligned}
\text{R.C.} &= \frac{\alpha}{2\pi}\left[\frac{25}{4} - \pi^2 + \frac{m_e^2}{m_\mu^2}\left(48\ln\frac{m_\mu}{m_e} - 18 - 8\pi^2\right)\right] \\
&\quad \times \left[1 + \frac{\alpha}{\pi}\left(\frac{2}{3}\ln\frac{m_\mu}{m_e} - 3.7\right) + \frac{\alpha^2}{\pi^2}\left(\frac{4}{9}\ln^2\frac{m_\mu}{m_e} - 2.0\ln\frac{m_\mu}{m_e} + C\right) + \ldots\right] \\
\alpha^{-1} &= 137.03599959(40) \qquad (3)
\end{aligned}
$$

where C corresponds to unknown non-logarithmic $\mathcal{O}(\alpha^3/\pi^3) \sim 10^{-8}$ corrections which are assumed to be insignificant. Employing (1,2,3) leads to

$$G_\mu = 1.16637(1) \times 10^{-5}\ \text{GeV}^{-2}. \qquad (4)$$

The Fermi constant can be compared with other precise measurements such as α, m_Z, $\sin^2\theta_W$, m_W, etc., and used to test the consistency of the Standard Model at the quantum loop level. For example, $G_\mu = \pi\alpha/\sqrt{2}m_W^2(1 - m_W^2/m_Z^2)(1 - \Delta r)$ where $\Delta r \simeq 0.0358$ (for $m_{\text{top}} = 174.3$ GeV and Higgs mass $m_H \simeq 125$ GeV) represents calculable electroweak radiative corrections [7]. In that role, G_μ helped predict the top quark's mass before its discovery and currently constrains the Higgs mass to relatively low preferred values $m_H \lesssim 220$ GeV. In addition, it provides a sensitive probe of "new physics" such as SUSY, Technicolor, Extra Dimensions etc. [5].

Recently, there have been several proposals [8–10] to further improve the measurement of τ_μ (and thereby G_μ) by as much as a factor of 20, bringing its uncertainty down to an incredible ± 1 ppm. Of course, such an improvement can only by fully utilized if the other electroweak parameters with which it is compared reach a similar level of precision and radiative corrections to their relationship (e.g. Δr) are computed at least through 2 loops. That confluence of advances appears unlikely in the foreseeable future. Nevertheless, given the fundamental nature and importance of G_μ, efforts to improve its determination should be strongly encouraged and pushed as far as possible. In that spirit, we examine in this paper several theoretical concerns that must be addressed in any ± 1 ppm study of τ_μ.

Precision muon lifetime experiments involve stopping μ^+ in material. At some level, the medium in which it comes to rest will affect the muon decay rate. The most straightforward issue to consider is the formation of muonium ($M = \mu^+e^-$) and its bound state effect on τ_μ.

During the slowing down of μ^+ in matter, muonium will form and be ionized many times [11,12]. Eventually it comes to (thermal) rest at which point the muon or muonium atom has a room temperature kinetic energy ~ 0.04 eV or thermal velocity $\beta_\mu^{\text{thermal}} \simeq 2.7 \times 10^{-5}$.

The actual fraction of stopped μ^+ that end up as muonium and decay while in that bound state is very medium dependent. It can range from a small fraction in metals to nearly 100% in some materials. If a significant modification of τ_μ occurred in muonium, a correction would have to be applied. For example, a published study [13] has claimed a 0.999516 reduction factor for the bound state decay rate. Such a large -484 ppm shift would be difficult to correct for at the ± 1 ppm level unless the muonium formation fraction was very precisely known. It would also impact the interpretation of existing τ_μ measurements [14] and might indicate the possibility of other large medium effects.

Given its potential importance, we have reexamined the muonium bound state effect on the muon lifetime. As we shall show, the leading correction turns out to be remarkably small, $\frac{\alpha^2 m_e^2}{2m_\mu^2} \simeq 6 \times 10^{-10}$ and easy to calculate.

We begin by recalling some basic properties of muonium [15]. It is a μ^+e^- Coulombic bound state with hydrogenic features. The reduced mass

$$m = \frac{m_e m_\mu}{m_\mu + m_e} \simeq 0.995 m_e \tag{5}$$

is slightly below the electron mass, but the difference is inconsequential for the considerations here and can be neglected. The energy levels are given by

$$E_n = -\frac{\alpha^2 m}{2n^2} = -13.5\,\text{eV}/n^2, \qquad n = 1, 2\ldots \tag{6}$$

For the $n = 1$ ground state the virial theorem tells us that the average (electron) kinetic energy is

$$\langle T \rangle = -E_1 = 13.5\,\text{eV} \tag{7}$$

while the average bound state potential energy is

$$\langle V \rangle = E_1 - \langle T \rangle = \alpha^2 m = -27.0\,\text{eV}. \tag{8}$$

Also, the $n = 1$ momentum distribution is given by [16]

$$4\pi p^2 \left|\psi(p)\right|^2 = \frac{32}{\pi}\alpha m \left(\frac{\alpha m}{p^2 + \alpha^2 m^2}\right)^4 p^2. \tag{9}$$

It peaks at $p^2 = \alpha^2 m^2/3$ which is somewhat below the average $\langle p^2 \rangle = \alpha^2 m^2$. In that configuration the e^- and μ^+ have equal but opposite momenta and r.m.s. velocities, $\beta_{\text{r.m.s.}} \equiv \langle \beta^2 \rangle^{1/2}$, $\beta_e = \alpha \simeq 1/137$, and $\beta_\mu \simeq \alpha m_e/m_\mu \simeq 3.5 \times 10^{-5}$. Note that the muon's r.m.s. bound state velocity is similar (somewhat larger) in magnitude to its room temperature thermal velocity.

Of course, (9) represents a non-relativistic approximation and should not be used indiscriminately for large p. However, it is adequate for the analysis presented here. In fig. 1, we display the bound state momentum probability distribution corresponding to (9).

Muonium bound state modifications of the muon decay spectrum and lifetime must vanish as α or $m_e/m_\mu \to 0$. It is, therefore, useful to organize such corrections as an $\alpha^n (m_e/m_\mu)^m$, $n, m = 1, 2\ldots$ expansion in the small parameters $\alpha \simeq 1/137$ and $m_e/m_\mu \simeq 1/207$. Terms of order $n + m > 4$ are completely negligible and can be safely ignored.

The leading $\mathcal{O}(\alpha m_e/m_\mu)$ effects result from the muon's non-zero velocity in the muonium rest frame. (For this discussion we take the muonium atom to be at rest in the lab frame.) The ground state velocity, $\beta_\mu \sim \alpha m_e/m_\mu$, will modify the e^+ decay spectrum relative to its shape for free μ^+ decay at rest (see fig. 2). For a given β_μ, it gives rise to Doppler broadening which corresponds to an overall dilation along with smearing of the positron energy over the range $(E_{e^+} \pm \beta_\mu p_{e^+})/\sqrt{1 - \beta_\mu^2}$. That distortion amounts to about ± 1.9 keV near the free decay endpoint $E_{e^+} = (m_\mu^2 + m_e^2)/2m_\mu$ for $\beta_\mu \sim \alpha m_e/m_\mu$. Such an effect leads to a small spectral tail beyond the usual endpoint. Of course, that enhancement is accompanied by a depletion below the usual endpoint. In fact, smearing corresponds primarily to a redistribution of the spectrum. Lorentz invariance requires that such effects cancel in the total integrated decay rate up to an overall time dilation factor $\gamma = 1/\sqrt{1-\beta_\mu^2}$ which slightly reduces the decay rate by $1/\gamma$. Averaging over the momentum distribution in Eq. (9) or its relativistic generalization replaces β_μ^2 by $\langle\beta_\mu^2\rangle \simeq \alpha^2 m_e^2/m_\mu^2$ and one finds [17]

$$\Gamma(M \to e^- e^+ \nu_e \bar{\nu}_\mu) = \left(1 - \frac{\langle\beta_\mu^2\rangle}{2}\right) \Gamma(\mu^+ \to e^+ \nu_e \bar{\nu}_\mu), \tag{10}$$

$$\tau_M = \tau_\mu \left(1 + \frac{\alpha^2 m_e^2}{2m_\mu^2}\right), \tag{11}$$

$$\frac{\tau_M - \tau_\mu}{\tau_\mu} = \frac{\alpha^2 m_e^2}{2m_\mu^2} \simeq 6 \times 10^{-10}. \tag{12}$$

Although the bound state velocity effects on the spectrum are relatively large $\mathcal{O}(\alpha m_e/m_\mu) \simeq$ 35ppm compared to the ± 1 ppm experimental lifetime goal, the overall shift in Eq. (12) is completely negligible. So, lifetime counting experiments that are independent of spectrum shape details are very insensitive to small muon bound state velocity effects.

In the case of muonium stopped in matter, the bound state and thermal muonium velocities should be approximately added in quadrature

$$\left(\beta_\mu^{\text{total}}\right)^2 \simeq \left(\beta_\mu^M\right)^2 + \left(\beta_\mu^{\text{thermal}}\right)^2. \tag{13}$$

At room temperature, $\beta_\mu^{\text{thermal}} \simeq 0.8\beta_\mu^M$ and the total time dilation shift in Eq. (12) increases by about a factor of 1.6. Nevertheless, it remains negligible at the 1 ppm level.

The muonium bound state Coulomb potential, $\langle V\rangle \sim -\alpha^2 m_e$, also modifies the positron decay spectrum. Those $\mathcal{O}(\alpha^2 m_e/m_\mu)$ corrections correspond to ~ -27 eV modifications of the spectrum. If they were to also affect the muon lifetime at that order, the effect would be larger than time dilation and possibly near the ± 1 ppm goal of future experiments. However, as we show in the Appendices, the leading effects cancel, leaving behind contributions that are much smaller than 1 ppm.

Aside from Doppler broadening, the bound state Coulomb interaction has two main effects on muon decay: phase-space suppression, because the muon is off mass shell, and final-state e^+e^- interactions after the decay. In fact these two effects largely cancel, because of charge conservation. We present a rigorous proof of this cancelation in the Appendices, but the nature of the cancelation is easily understood if we replace muonium by a simpler system, where the muon is bound to a static charge distribution with density $\rho(r) \equiv e|\psi(r)|^2$, rather than to an electron.

Initially the muon in our simplified system is at the center of the charge distribution with negligible kinetic energy and potential energy $-V_0$, where $V_0 = 27\,\text{eV}$. Since the muon is below mass shell, the maximum energy allowed to the positron after the muon decays is lowered by 27 eV from what it is for free-muon decay. This phase-space contraction tends to lower the decay rate. The positron, however, is formed at the center of our "muonium" atom and, because it has the same charge as the muon, it initially has the same potential energy, $-V_0$, by virtue of its interactions with the charge distribution $\rho(r)$. The positron rapidly leaves the atom, but its asymptotic kinetic energy is reduced from its initial kinetic energy by the 27 eV needed for it to climb out of the potential well due to $\rho(r)$. Thus the entire positron energy distribution is shifted down by $V_0 = 27\,\mathrm{eV}$:

$$\frac{\mathrm{d}\Gamma_M(E)}{\mathrm{d}E_{e^+}} = \frac{\mathrm{d}\Gamma_\mu(E+V_0)}{\mathrm{d}E_{e^+}} \tag{14}$$

This is the effect of final-state interactions. The two effects, phase-space contraction and final-state interactions, cancel in the total decay rate because the energy distribution and the kinematic limit are shifted by the same amount, leading to the same total integral for the decay rate:

$$\begin{aligned}\int^{m_\mu/2-V_0} \frac{\mathrm{d}\Gamma_M(E)}{\mathrm{d}E_{e^+}}\,\mathrm{d}E &= \int^{m_\mu/2-V_0} \frac{\mathrm{d}\Gamma_\mu(E+V_0)}{\mathrm{d}E_{e^+}}\,\mathrm{d}E \\ &= \int^{m_\mu/2} \frac{\mathrm{d}\Gamma_\mu(E')}{\mathrm{d}E_{e^+}}\,\mathrm{d}E' \end{aligned} \tag{15}$$

At very low-energies, the positron can bind to the charge distribution and our analysis must be modified. However any such modification is of order $\alpha^3 m_e^3/m_\mu^3 \ll \alpha^2 m_e^2/m_\mu^2$ (the time dilation effect) for the total decay rate since the spectrum is highly suppressed, by phase space $\sim pE\mathrm{d}E$, in that low-energy region (see fig. 2). Such corrections are negligible.

Other effects, e.g. hyperfine interactions, e^+e^- hard scattering etc., are also of higher order in α and/or m_e/m_μ and therefore suppressed. Thus, overall, the time dilation effects in (10)–(12) represent the dominant bound state correction to the muonium lifetime.

The cancelation of $\mathcal{O}(\alpha^2 m_e/m_\mu)$ effects in the total μ^+ decay rate is analogous to other cancelations of a similar nature that have been previously pointed out.

A recent example of that phenomenon has been encountered in inclusive B decays. The total decay rate of a meson or baryon containing a heavy b quark can be approximated by a "free" b quark decay rate. If the rate is expressed in terms of a short distance b quark mass, corrections to that approximation are $\mathcal{O}(1/m_b^2)$ rather than $\mathcal{O}(1/m_b)$. They stem from time dilation, just as we have found for the much simpler muonium bound state, as well as from hyperfine interactions (which for muonium are suppressed by α^4). In that case, the absence of $1/m_b$ corrections is due to color conservation [18]. Our NRQED analysis in the Appendix is closely related to the standard B analysis.

Another example of the cancelation of V dependence in total decay rates was found by H. Überall [19] in his classic study of muonic atoms, μ^-–Nuclei bound states. There, the μ^- is bound to the nucleus by a much stronger $V \sim -Z^2\alpha^2 m_\mu$ than muonium. Nevertheless, Überall found that the Coulomb potential reduction of initial state phase space was canceled by final state e^-–Nuclei Coulombic interactions in the total decay rate. In fact, he also found for that example that the leading bound state effect was time dilation which reduces the effective μ^- decay rate in muonic atoms by

$$\gamma^{-1} = \sqrt{1 - Z^2\alpha^2} \simeq 1 - \frac{1}{2}Z^2\alpha^2, \qquad (Z^2\alpha^2 \ll 1). \tag{16}$$

Of course, besides ordinary decay $\mu^- \to e^-\bar{\nu}_e\nu_\mu$, the muonic atom can undergo weak capture, $\mu^- p \to \nu_\mu n$, with protons in the nucleus. In fact, for heavy nuclei with $Z \gtrsim 12$ the capture process dominates over ordinary decay. Capture rates are generally obtained by measuring the μ^- lifetime in a material and subtracting out the ordinary decay rate ($\mu^- \to e^-\bar{\nu}_e\nu_\mu$) using the correction in (16). That prescription needs some adjustment for high Z nuclei where residual low energy e^-−Nuclei interactions can significantly suppress the decay rate and become comparable to time dilation [20].

The analog of $\mu^- p$ capture for muonium is annihilation $M \to \nu_e\bar{\nu}_\mu$. We have computed that rate for the $n = 1$ ground state and find [21,22]

$$\Gamma(M \to \nu_e\bar{\nu}_\mu) = 48\pi \left(\frac{\alpha m_e}{m_\mu}\right)^3 \Gamma(\mu^+ \to e^+\nu_e\bar{\nu}_\mu)$$
$$\simeq 6.6 \times 10^{-12}\, \Gamma(\mu^+ \to \text{all}). \tag{17}$$

Our result is in accord with the more general analysis of Ref. [23]. That rate is of higher order in $\alpha m_e/m_\mu$ than the time dilation correction in (11) and 100 times smaller. It can be safely neglected.

The very small $\mu^- p \to \nu_\mu n$ capture rate for protons has been determined by comparing measurements of τ_{μ^+} and τ_{μ^-} in liquid hydrogen [24]. In that way, the induced weak pseudoscalar coupling g_p of the nucleon can be inferred. Our general analysis along with Überall's study lend theoretical support to this method. Increasing the precision of those measurements by a factor of 10–20 (if possible) could provide an interesting confrontation with theory [25].

Our analysis of the muonium bound state effect on the μ^+ lifetime provides a simple way of estimating other effects of a stopping medium. For example, in metals the fraction of stopped μ^+ in muonium is tiny. Instead, the thermal muon is screened by conduction band electrons. The density of screening electrons near the μ^+ is expected to be comparable to muonium, resulting in a similar V and average μ^+ velocity. Hence, the distortion of the e^+ emission spectrum due to Doppler smearing and the final e^+ potential energy should be roughly the same as muonium. Due to Lorentz and gauge invariance, the leading effects will continue to cancel in the total decay rate. The main medium effect on the μ^+ lifetime will again be time dilation due to $\beta_\mu \neq 0$ (from thermal and electromagnetic interactions) which will be negligible $\sim \mathcal{O}(10^{-9})$. In the Appendix, we describe how such medium effects can be analyzed using nonrelativistic QED.

A possibility that we need only briefly discuss is the effect of the medium on radiative muon decay. Ordinary free muon decay bremsstrahlung, $\mu^+ \to e^+\nu_e\bar{\nu}_\mu\gamma$ (as well as $\gamma\gamma$ and e^+e^- production), is included in the $\mathcal{O}(\alpha)$ and $\mathcal{O}(\alpha^2)$ terms of Eq. (3). Those effects are rather small, despite the fact that soft and collinear bremsstrahlung can significantly modify the spectrum. As in our above analysis, those large logarithmic infrared effects primarily give rise to a redistribution of the spectrum but tend to cancel in the total decay rate [26,27].

Our general arguments carry over to radiative muonium decay. The radiative muonium decay spectrum will be distorted by Doppler smearing and V dependence, but those effects will largely cancel in the total decay rate, again leaving time dilation as the primary correction. A similar argument can be applied to alternative medium potential effects such as screening in metals.

Other long distance properties of a stopping medium may be more subtle to analyze. For example, after the μ^+ decay process, an outgoing e^+ might emit outer bremsstrahlung or Čerenkov radiation due to its material environment. Those effects will modify the e^+

and electromagnetic emission spectra. However, such changes again correspond to a redistribution of events and should cancel (to a very good approximation) in the total decay rate or lifetime (assuming Lorentz and gauge invariance are maintained in the analysis). The space-time scale associated with the actual decay process (not the lifetime) is $1/m_\mu$. That space-time interval is much too small to be affected by long distance environmental conditions.

In summary we have found that lifetimes of muonium and a free μ^+ differ in leading order only by time dilation effects which are $\sim 10^{-9}$ and completely negligible for future ± 1 ppm experiments. The smallness of that difference follows from Lorentz and gauge invariance which guarantee cancelations among considerably larger spectrum distortions. A similar suppression of other medium effects on the μ^+ lifetime is also expected.

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APPENDIX A: BETHE-SALPETER AND NRQED ANALYZES

In the text, we described how spectral shifts occur in muonium decay but largely cancel in the total decay rate, leaving time dilation as the primary bound state correction. Here, a proof of those cancelations is given using a Bethe-Salpeter bound state approach. A second formalism, nonrelativistic QED (NRQED) [28], is then employed to illustrate the important role of electromagnetic gauge invariance or charge conservation in the cancelation. That method provides a powerful means of parameterizing other more general medium effects and showing that they also lead to negligible corrections to the muon lifetime in matter.

1. Bethe-Salpeter Analysis

We can compute the leading bound state corrections to muonium decay using standard Bethe-Salpeter perturbation theory for the bound state energies [16]. The decay rate is obtained from the imaginary part of the energy ($\Gamma = -2\mathrm{Im}\, E$). The leading effect comes from the $\mu^+ \to e^+\nu\bar{\nu} \to \mu^+$ contribution $\Sigma_{e^+\nu\bar{\nu}}(p)$ to the muon's self energy (see figs. 3 and 4(a)). Since the decay products are typically highly relativistic, we can Taylor expand the self energy about the mass-shell momentum in the usual fashion [29]:

$$\mathrm{Im}\,\Sigma_{e^+\nu\bar{\nu}}(p) = -\frac{\Gamma_\mu}{2} + (p\cdot\gamma - m_\mu)\,\delta Z_{e^+\nu\bar{\nu}} + \cdots \tag{A1}$$

where Γ_μ is the free muon decay rate and m_μ is the muon pole mass. In muonium, the second term in this expansion gives the leading decay rate correction due to the bound state reduction of the $e^+\nu\bar{\nu}$ phase space. The shift in the bound state energy due to this self energy is, in first order perturbation theory,

$$\begin{aligned}\overline{\psi}\,(p_e\cdot\gamma-m_e)\,\mathrm{Im}\,\Sigma_{e^+\nu\bar{\nu}}(p_\mu)\,\psi &= -\frac{\Gamma_\mu}{2}\,\overline{\psi}(p_e\cdot\gamma-m_e)\psi \\ &\quad+\delta Z_{e^+\nu\bar{\nu}}\,\overline{\psi}(p_e\cdot\gamma-m_e)(p_\mu\cdot\gamma_{(\mu)}-m_\mu)\psi,\end{aligned}\tag{A2}$$

where ψ is the μ^+e^- wave function, $\gamma_{(\mu)}$ denotes Dirac matrices for the muon spinor part of ψ, p_e and p_μ are the electron and muon momenta respectively, and integration over the wave function's momenta distribution is implicit. The constant $\delta Z_{e^+\nu\bar{\nu}}$ is of order Γ_μ/m_μ.

The coefficient of $-\Gamma_\mu/2$ in this expectation value is nearly equal to unity since the wave functions are normalized such that [30]

$$\overline{\psi}\,\gamma^0_{(\mu)}\,(p_e\cdot\gamma-m_e)\,\psi=1.\tag{A3}$$

The standard nonrelativistic expansion

$$\frac{\overline{u}(p)1u(p)}{\overline{u}(p)\gamma^0u(p)}\approx 1-\frac{p^2}{2m^2}\tag{A4}$$

implies, therefore, that

$$\overline{\psi}\,(p_e\cdot\gamma-m_e)\,\psi\approx\langle 1-\beta_\mu^2/2\rangle\tag{A5}$$

where β_μ is the muon velocity. This factor decreases the decay rate for muonium; it is the time dilation caused by the muon's motion within the atom.

The second term in expectation value (A2) is simplified by noting that

$$(p_e\cdot\gamma-m_e)(p_\mu\cdot\gamma_{(\mu)}-m_\mu)\,\psi=V\,\psi,\tag{A6}$$

from the Bethe-Salpeter equation, where V is the binding potential (dominated by the Coulomb interaction). Thus, through order α^2,

$$\overline{\psi}\,(p_e\cdot\gamma-m_e)\,\mathrm{Im}\,\Sigma_{e^+\nu\bar{\nu}}(p_\mu)\,\psi\;\approx\;-\frac{\Gamma_\mu}{2}\,\langle 1-\beta_\mu^2/2\rangle+\delta Z_{e^+\nu\bar{\nu}}\,\langle V\rangle,\tag{A7}$$

where the first correction to the free decay rate is due to time dilation, and the second correction is due to the contraction of final-state phase space. These two corrections are of relative orders $\alpha^2 m_e^2/m_\mu^2$ and $\alpha^2 m_e/m_\mu$ respectively.

A second contribution is relevant in order α^2. This is from the imaginary part of the muon-photon vertex correction $\Gamma^\rho_{e^+\nu\bar{\nu}}$ due to the fluctuation $\mu^+\to e^+\nu\bar{\nu}\to\mu^+$ — that is, the correction obtained by attaching a photon to the e^+ in the muon self-energy diagram for $\Sigma_{e^+\nu\bar{\nu}}$ (see fig. 4(b)). Physically this corresponds to a final-state interaction between the outgoing positron (from the decay) and the electron (from the atom). The momentum transfer carried by atomic photons ($\approx\alpha m_e$) is tiny compared to the typical $e^+\nu\bar{\nu}$ momenta, and so we can Taylor expand the vertex function to obtain the leading contribution:

$$\mathrm{Im}\,\Gamma^\rho_{e^+\nu\bar{\nu}}=-\delta Z_{e^+\nu\bar{\nu}}\,\gamma^\rho+\cdots,\tag{A8}$$

where the remaining terms on the right all vanish on mass shell for zero momentum transfer. The constant $\delta Z_{e^+\nu\bar{\nu}}$ here is the same as that appearing in the self energy; this is required by QED gauge invariance and can be proven using Ward identities in the standard fashion. This vertex correction results in a perturbation given by

$$-\,\delta Z_{e^+\nu\bar{\nu}}\,\overline{\psi}V\psi \;=\; -\delta Z_{e^+\nu\bar{\nu}}\,\langle V\rangle \tag{A9}$$

through order α^4 corrections (due to higher order terms in the expansion (A8) of the vertex correction). Remarkably, this final-state interaction completely cancels the part of the self-energy correction due to the contraction of phase space. Thus, through order $\alpha^2 m_e^2/m_\mu^2$ the muonium decay rate is

$$\Gamma_{\mu e} = \Gamma_\mu\,\langle 1-\beta_\mu^2/2\rangle. \tag{A10}$$

It should be emphasized that the cancelation of final-state and phase-space effects is a particular consequence of QED gauge invariance. Had photons been massless scalars instead of vector gauge bosons, for example, the final-state interaction would have vanished and the contraction of phase space would have been the dominant decay rate correction.

2. NRQED and Muon Decay

The Bethe-Salpeter formalism is awkward for analyzing muonium or other medium effects on muon decay with greater precision. A better approach follows from the observation that the fluctuation $\mu^+ \to e^+\nu\bar{\nu} \to \mu^+$ occurs over distances, of order $1/m_\mu$, which are tiny compared with atomic length scales. Thus, the effects of such fluctuations can be modeled in a low-energy effective Lagrangian by local, gauge-invariant interactions. A standard non-relativistic effective field theory used in high-precision studies of muonium is nonrelativistic QED (NRQED). The corrections due to muon decay, being local, are easily included in that formalism as (nonunitary) corrections to the standard NRQED Lagrangian:

$$\begin{aligned}\mathcal{L}_{\rm NRQED} = \psi_\mu^\dagger\Bigg\{ & iD_t + \frac{\mathbf{D}^2}{2m_\mu} + \cdots \\ & + \frac{i\Gamma_\mu}{2}\Bigg(1 + c_1\frac{\mathbf{D}^2}{2m_\mu^2} - c_2\frac{\mathbf{D}^4}{8m_\mu^4} + \cdots \\ & \qquad + d_1\frac{\psi_e^\dagger\psi_e}{m_\mu^3} + d_2\frac{\sigma\cdot\psi_e^\dagger\sigma\psi_e}{m_\mu^3} + \cdots \\ & \qquad + f_1\frac{e\sigma\cdot\mathbf{B}}{m_\mu^2} + f_2\frac{e\nabla\cdot\mathbf{E}}{m_\mu^3} + \cdots\Bigg)\Bigg\}\psi_\mu.\end{aligned} \tag{A11}$$

Here D_t and $\mathbf{D}$ are the temporal and spatial gauge-covariant derivatives, $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic field operators, and ψ_μ and ψ_e are two-component Pauli fields for muons and electrons. This effective theory provides a rigorous, systematic framework for analyzing nonrelativistic muons, including relativistic corrections as well as muonium and medium effects on muon decay, to whatever precision is desired.

The couplings c_1, c_2, $d_1\ldots$ in the effective Lagrangian are all dimensionless and have perturbative expansions in α. The c_i's are determined by computing muon decay, using the effective theory, for muons in different reference frames. They are tuned until the decay rate predicted by the effective theory agrees with the relativistic result

$$\Gamma_\mu(\beta_\mu) = \Gamma_\mu(0)\left(1 - \frac{\beta_\mu^2}{2} - \frac{\beta_\mu^4}{8} - \cdots\right); \tag{A12}$$

at tree-level $c_1 = c_2 = 1$. The d_i's incorporate the contribution from muon-electron annihilation: $\mu e \to \nu\bar{\nu}$ (see Eq. (17)). The f_i's are computed by comparing the on-shell, renormalized vertex function $\Gamma_{e^+\nu\bar{\nu}}$, expanded in powers of the external momenta, with predictions from the effective theory.

Two features of the effective theory deserve comment. One is that there are no terms of the form

$$\frac{i\Gamma_\mu}{m_\mu}\,\psi_\mu^\dagger e A^0 \psi_\mu, \tag{A13}$$

where A^0 is the scalar potential in electromagnetism (that is, the Coulomb potential in an atom). Such a term would be required to generate a contribution such as that due to either phase-space contraction (second term in Eq. (A7)) or final state e^+e^- interactions alone. It cannot arise here because it would violate QED gauge invariance. This confirms the detailed Bethe-Salpeter analysis of the previous section which demonstrated the cancelation of such terms.

The second observation is that the parameter Γ_μ appearing in the effective Lagrangian is precisely the free muon decay rate, to all orders in α; it is *not* a "bare" decay rate that is further renormalized by medium dependent radiative corrections. This is because the operator $\psi_\mu^\dagger\psi_\mu$ in

$$\delta\mathcal{L} \equiv \frac{i\Gamma_\mu}{2}\,\psi_\mu^\dagger\psi_\mu \tag{A14}$$

is a conserved quantity in the nonrelativistic theory; it is the pure number operator that counts muons and so cannot be renormalized. Consequently

$$\frac{\langle\phi|\delta\mathcal{L}|\phi\rangle}{\langle\phi|\phi\rangle} = \frac{i\Gamma_\mu}{2} \tag{A15}$$

for any state $|\phi\rangle$ that contains a single muon — including muonium states as well as arbitrarily complicated many-electron states or other medium effects that describe muons inside bulk materials. This "no-renormalization theorem" implies that the only corrections to the free muon decay rate in such systems come from the explicit nonunitary, factorizable correction terms in the effective Lagrangian, the terms with couplings c_i, d_i, $f_i\ldots$ that are directly related to muon decay. As discussed in the text, these corrections are typically of order 1 ppb or smaller. Other effects not of this form, due to external photons or multi-electron excitations, cannot modify the total decay rate.

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[30] The right-hand side of the normalization condition would be the electromagnetic form factor of muonium at zero momentum transfer if the electron had no electric charge, and provided that the potential is instantaneous (which is true up to $\mathcal{O}(\alpha^3)$ corrections which we ignore). Charge conservation or gauge invariance demands that this form factor equal unity.

FIGURES

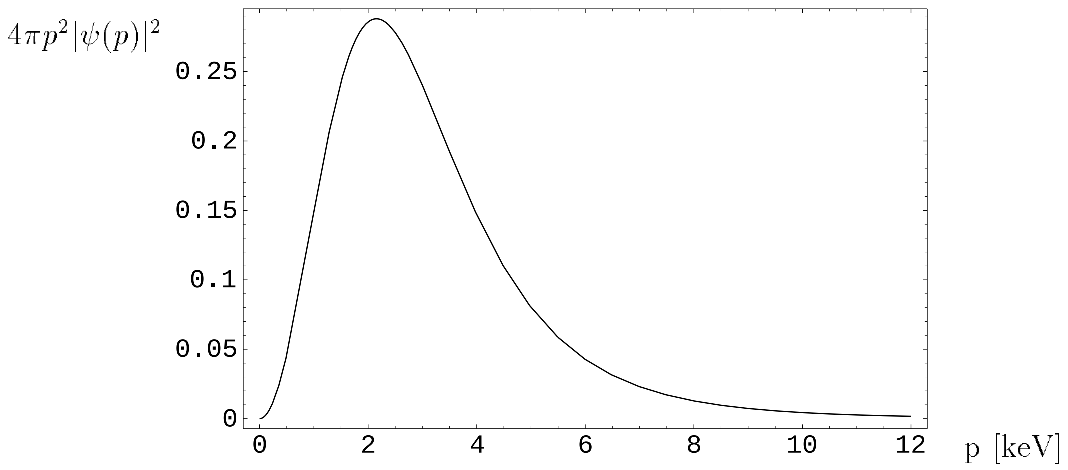

FIG. 1. Momentum probability distribution in muonium $1S$ state.

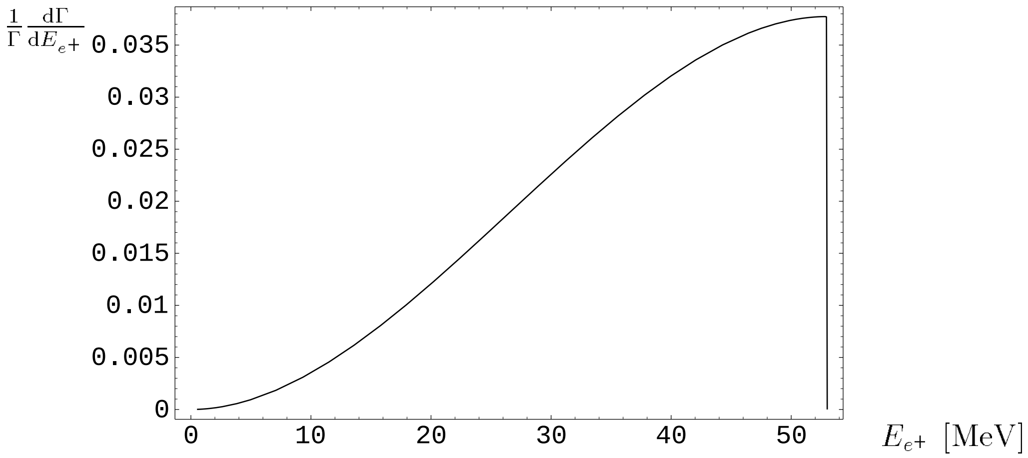

FIG. 2. Positron energy spectrum for free μ^+ decay at rest.

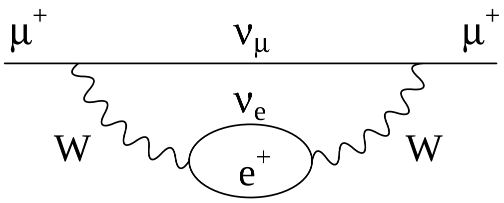

FIG. 3. Self energy of the muon. Its imaginary part, computed in the external Coulombic field, determines the lowest order muonium width.

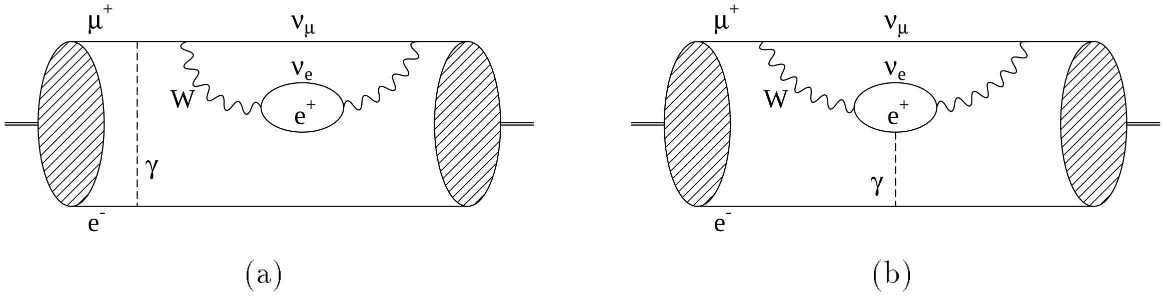

FIG. 4. Coulomb interactions in muonium decay.