Considerations Concerning the Radiative Corrections to Muon Decay in the Fermi and Standard Theories

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Abstract

The FAC, PMS and BLM optimization methods are applied to the QED corrections to the muon lifetime in the Fermi V-A theory. The FAC and PMS scales are close to m_e, while the BLM scale nearly concides with the geometric average $\sqrt{m_e m_\mu}$. The optimized expressions are employed to estimate the third order coefficient in the $\alpha(m_\mu)$ expansion and the theoretical error of the perturbative series. Using arguments based on effective field theory and a simple examination of Feynman diagrams, it is shown that, if contributions of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$ are neglected, the corrections to muon decay in the SM factorize into the QED correction of the Fermi V-A theory and the electroweak amplitude $g^2/(1-\Delta r)$, both of which are strictly scale-independent. We use the results to clarify how the QED corrections to muon decay and the Fermi constant G_F should be used in the SM, and what is the natural choice of scales if running couplings are employed.

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I. INTRODUCTION

The one-loop radiative corrections to μ-decay in the Fermi theory were evaluated approximately four decades ago. These studies included the correction to the electron spectrum and muon lifetime τ_μ [1,2], as well as the momentum dependence of the asymmetry and the integrated asymmetry for polarized muons [1]. Although the Fermi theory is, in general, non-renormalizable, the μ-decay case is especial. For vector and axial vector interactions of the charge retention order, which are the relevant couplings in the two-component theory of the neutrino and in the V-A theory, a theorem assures us that, to first order in G_μ, but all orders in α, the corrections to μ-decay are convergent after mass and charge renormalization [3]. A striking cancellation of mass singularities in the correction to the lifetime and integrated asymmetries was found, which occurs also for the scalar, pseudoscalar, and tensor Fermi interactions, and the β-decay lifetime [1]. These observations were one of the motivations in the derivation of the KLN theorem [4,5]. Years later, in order to clarify a controversy that arose in the determination of the Fermi constant, the corrections of $\mathcal{O}\left(\alpha^2 \ln\left(m_\mu/m_e\right)\right)$ to τ_μ were obtained [6].

Very recently, in an important theoretical development, van Ritbergen and Stuart completed the evaluation of $\mathcal{O}\left(\alpha^2\right)$ corrections to τ_μ in the local V-A theory, in the limit $m_e^2/m_\mu^2 \to 0$ [7,8]. Their final answer can be expressed succintly as

$$C\left(m_\mu\right) = \frac{\alpha(m_\mu)}{\pi} c_1 + \left(\frac{\alpha(m_\mu)}{\pi}\right)^2 c_2, \tag{1}$$

$$c_1 = \frac{1}{2}\left(\frac{25}{4} - \pi^2\right); c_2 = 6.700 \ . \tag{2}$$

In Eq.(1), $\alpha(m_\mu) c_1/\pi$ is the one-loop result [1] and $\alpha(m_\mu)$ is a running coupling defined by

$$\alpha(m_\mu) = \frac{\alpha}{1 - \left(\frac{2\alpha}{3\pi} + \frac{\alpha^2}{2\pi^2}\right) \ln \frac{m_\mu}{m_e}}. \tag{3}$$

In Refs. [7,8], the contribution of the last term in the denominator of Eq.(3) is separated out, as $(\alpha^3/2\pi^2)\ln\left(m_\mu/m_e\right)$. The two expressions differ by only 0.2 ppm, so that we will employ the more succint expression of Eq.(3).

In Section 2, we apply the traditional FAC [9], PMS [10], and BLM [11] optimization schemes to the expansion of Eq.(1). We use those methods for estimations of the third order coefficient and the theoretical error in Eq.(1). In Section 3 we turn our attention to the radiative corrections to muon decay in the Standard Model (SM). Using arguments based on effective field theory and simple considerations of Feynman diagrams, we show that, if terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$ are neglected, the radiative corrections factor out into $[1 + C(m_e)]$ and the electroweak amplitude $g^2/(1 - \Delta r)$, which are separately scale-independent. We use the results to discuss the application of the Fermi constant G_F in electroweak physics. Section 4 presents our conclusions.

II. APPLICATION OF OPTIMIZATION METHODS

As it is well known, $\alpha(\mu)$ does not run below the m_e scale and $\alpha(m_e)$ is identified with the conventional fine structure constant, $\alpha^{-1} = 137.03599959(38)(13)$ [12]. Setting then $\alpha = \alpha(m_e)$ in Eq.(3) and replacing $m_e \to \mu$, we have

$$\alpha(m_\mu) = \frac{\alpha(\mu)}{1 - \left(\frac{2\alpha(\mu)}{3\pi} + \frac{\alpha^2(\mu)}{2\pi^2}\right) \ln \frac{m_\mu}{\mu}}. \tag{4}$$

Inserting this expression into Eq.(1), expanding and truncating in second order, we find

$$C(\mu) = \frac{\alpha(\mu)}{\pi} c_1 + \left(\frac{\alpha(\mu)}{\pi}\right)^2 \left[c_2 + \frac{2}{3} c_1 \ln \frac{m_\mu}{\mu}\right]. \tag{5}$$

The method of Fastest Apparent Convergence (FAC) [9] chooses the scale $\mu_{\rm FAC}$ in such a manner that the NLO term vanishes. Thus, we have $\ln(m_\mu/\mu_{\rm FAC}) = -3c_2/(2c_1)$, which leads to $\mu_{\rm FAC} = 0.801\ m_e$. As $\alpha(\mu)$ does not run below $\mu = m_e$, we interpret this result as $\mu_{\rm FAC} = m_e$. Eq.(5) becomes

$$C(\mu_{\rm FAC}) = C(m_e) = \frac{\alpha}{\pi} c_1 + \left(\frac{\alpha}{\pi}\right)^2 \left[c_2 + \frac{2}{3} c_1 \ln \frac{m_\mu}{m_e}\right], \tag{6}$$

an expansion in terms of the fine structure constant α. The $\mathcal{O}((\alpha/\pi)^2 ln(m_\mu/m_e))$ term concides with the expression found in Ref. [6]. Recalling $m_\mu/m_e = 206.768273(24)$ [12], we see that the two terms between square brackets nearly cancel each other and Eq.(6) becomes

$$C(\mu_{\rm FAC}) = C(m_e) = \frac{\alpha}{\pi} c_1 + \left(\frac{\alpha}{\pi}\right)^2 0.26724\ . \tag{7}$$

The Principle of Minimal Sensitivity (PMS) [10] identifies $\mu_{\rm PMS}$ with the stationary point of $C(\mu)$. Applying $\mu(d/d\mu)$ to $C(\mu)$, and recalling the renormalization group equation (RGE)

$$\mu \frac{d}{d\mu} \alpha(\mu) = \frac{2}{3\pi} \alpha^2(\mu) + \frac{\alpha^3(\mu)}{2\pi^2}, \tag{8}$$

we obtain

$$\mu \frac{d}{d\mu} C(\mu) = \frac{4}{3} \left(\frac{\alpha(\mu)}{\pi}\right)^3 \left[c_2 + \frac{3}{8} c_1 + \frac{2}{3} c_1 \ln \frac{m_\mu}{\mu}\right]. \tag{9}$$

Thus, the stationary point is given by

$$\ln \frac{\mu_{\rm PMS}}{m_\mu} = \frac{3}{2} \left(\frac{3}{8} + \frac{c_2}{c_1}\right),$$

which leads to $\mu_{\rm PMS} = 1.40636\ m_e$. We note that Eq.(8) is consistent with

$$\alpha(\mu) = \frac{\alpha}{1 - \left(\frac{2\alpha}{3\pi} + \frac{\alpha^2}{2\pi^2}\right) \ln \frac{\mu}{m_e}}, \tag{10}$$

an expression that can be obtained by replacing $m_\mu \to \mu$, $\mu \to m_e$ in Eq.(4), and will be useful in the following discussion. Inserting $\mu = \mu_{\rm PMS}$ in Eq.(5), we obtain

$$C(\mu_{\rm PMS}) = \frac{\alpha(\mu_{\rm PMS})c_1}{\pi} - \left(\frac{\alpha(\mu_{\rm PMS})}{\pi}\right)^2 \frac{3}{8}c_1, \tag{11}$$

or

$$C(\mu_{\rm PMS}) = \frac{\alpha(\mu_{\rm PMS})c_1}{\pi} + \left(\frac{\alpha(\mu_{\rm PMS})}{\pi}\right)^2 0.67868\ . \tag{12}$$

The BLM method [11] chooses $\mu_{\rm BLM}$ so as to cancel the terms in c_2 proportional to n_f, the number of light fermions. In the μ-decay case, there is only one light fermion, the electron. From Ref. [7] we learn that the electron loop contributions to c_2 (virtual loops and pair creation) is 3.22034. Splitting $c_2 = 3.22034 + 3.47966$, one chooses $\mu_{\rm BLM}$ to cancel the first contribution in the second order coefficient of Eq.(5). Thus, $\ln(m_\mu/\mu_{\rm BLM}) = -(3/2)(3.22034/c_1)$, which leads to $\mu_{\rm BLM} = m_\mu/14.4267 = 14.3323\ m_e$. We note that this is very close to $\sqrt{m_e m_\mu} = 14.38\ m_e$, the geometric average of the two masses. The BLM expansion is

$$C(\mu_{\rm BLM}) = \frac{\alpha(\mu_{\rm BLM})c_1}{\pi} + \left(\frac{\alpha(\mu_{\rm BLM})}{\pi}\right)^2 3.47966\ . \tag{13}$$

The FAC, PMS, and BLM optimization expressions of Eqs.(7, 12, 13), can be used to estimate the coefficient c_3 of $(\alpha(m_\mu)/\pi)^3$ in the expansion of Eq.(1). It is well known that such estimations are not particularly useful if contributions of an important class of new diagrams open up at the relevant level. For instance, in the radiative corrections to $g-2$, large contributions due to the light by light scattering open up already in $\mathcal{O}(\alpha^3)$. In μ-decay we encounter a more felicitous situation, as light by light scattering contributes only in $\mathcal{O}(\alpha^4)$. Writing the three optimized expansions in the generic form

$$C(\mu^*) = \frac{\alpha(\mu^*)}{\pi}c_1 + \left(\frac{\alpha(\mu^*)}{\pi}\right)^2 c_2^*, \tag{14}$$

and expressing $\alpha(\mu^*)$ in terms of $\alpha(m_\mu)$ by replacing $m_\mu \to \mu^*$, $\mu \to m_\mu$ in Eq.(4), we find the estimation

$$(c_3)_{est} = \frac{4}{9}c_1 \ln^2\frac{\mu^*}{m_\mu} + \left(\frac{c_1}{2} + \frac{4}{3}c_2^*\right)\ln\frac{\mu^*}{m_\mu}. \tag{15}$$

Using the values of μ^* and c_2^* obtained above, Eq.(15) leads to

$$(c_3)_{est} = \begin{cases} -19.9\ \text{(FAC)} \\ -20.0\ \text{(PMS)} \\ -15.7\ \text{(BLM)} \end{cases} \tag{16}$$

We see that the three estimates are quite close.

It is also interesting to compare the numerical results of the optimized expansions among themselves and with that of Eq.(1). Using the values of $\mu_{\rm PMS}$ and $\mu_{\rm BLM}$ found before, and evaluating $\alpha(\mu^*)$ ($\mu^* = \mu_{\rm PMS}$,$\mu_{\rm BLM}$) via Eq.(10), the expansions of Eqs.(7,12,13,1) lead to

$$
\begin{aligned}
C(m_e) &= -4.202402 \times 10^{-3} \quad (FAC), & (17)\\
C(\mu_{\rm PMS}) &= -4.202403 \times 10^{-3} \quad (PMS), & (18)\\
C(\mu_{\rm BLM}) &= -4.202348 \times 10^{-3} \quad (BLM), & (19)\\
C(m_\mu) &= -4.202147 \times 10^{-3} \quad (\alpha(m_\mu)\ \text{exp.}). & (20)
\end{aligned}
$$

We see that the three optimization methods give very close results, with a maximum absolute difference of 5.4×10^{-8}. As the important mass scales for μ decay in the Fermi theory are m_e and m_μ, it is natural to consider $m_e < \mu < m_\mu$ as the relevant range. We may therefore take the maximum difference among the four evaluations above as an estimate of the theoretical error. This is given by the difference $C(m_\mu) - C(\mu_{\rm PMS}) = 2.6 \times 10^{-7}$, which is equivalent to the estimated third order contribution $20.0[\alpha(m_\mu)/\pi]^3$ in the PMS estimate [Cf.Eq.(16)]. This leads to an error of 1.3×10^{-7} in the determination of G_F, which is very close to the estimate given in Ref. [8] from the consideration of the known leading third order logarithm in $C(m_e)$. We also recall that the one-loop corrections proportional to powers and logarithms of m_e^2/m_μ^2 can be obtained by combining Ref. [13] and Ref. [1], and have been reported in Refs. [8,14]. When the tree-level μ decay phase space is factored out, the leading correction of this type is

$$\frac{\alpha}{\pi}\frac{m_e^2}{m_\mu^2}\left[24 log\frac{m_\mu}{m_e} - 9 - 4\pi^2\right] = 4.3 \times 10^{-6}. \tag{21}$$

In very high precision calculations, Eq.(21) should be added to Eqs.(17-20) so that the expansions, rounded to four decimals, become

$$
\begin{aligned}
C(m_e) = C(\mu_{\rm PMS}) &= -4.1981 \times 10^{-3}, & (22)\\
C(\mu_{\rm BLM}) &= -4.1980 \times 10^{-3}, & (23)\\
C(m_\mu) &= -4.1978 \times 10^{-3}. & (24)
\end{aligned}
$$

The error due to the uncalculated terms of $\mathcal{O}((\alpha/\pi)^2(m_e^2/m_\mu^2)\ln^2 m_\mu/m_e)$ has been estimated to be a few times 10^{-7} [8]. This seems quite resonable. In fact, we note that if the $\mathcal{O}((\alpha/\pi)^2(m_e^2/m_\mu^2))$ contributions had the same magnitude relative to the leading $\mathcal{O}(\alpha^2)$ term $(\alpha(m_\mu)/\pi)^2 c_2$ in Eq.(1), as the $\mathcal{O}((\alpha/\pi)(m_e^2/m_\mu^2))$ bear with respect to $(\alpha(m_\mu)/\pi)c_1$, their contribution would be even smaller, a few times 10^{-8}.

III. FACTORIZATION OF THE RADIATIVE CORRECTIONS TO MUON DECAY IN THE SM

In the on-shell renormalization scheme of the SM, it is customary to write $1/\tau_\mu$ in the form

$$\frac{1}{\tau_\mu} = \frac{P}{32} \frac{g^4}{m_W^4} \frac{[1 + C(m_e)]}{(1 - \Delta r)^2}, \tag{25}$$

$$P = f\left(\frac{m_e^2}{m_\mu^2}\right)\left[1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2}\right]\frac{m_\mu^5}{192\pi^3}, \tag{26}$$

$$f(x) = 1 - 8x - 12x^2 \ln x + 8x^3 - x^4, \tag{27}$$

where $g^2 = e^2/\sin^2\theta_w$, $\sin^2\theta_w = 1 - M_W^2/M_Z^2$, M_W and M_Z are pole masses, $C(m_e)$ is the radiative correction of the Fermi V-A theory and Δr is the electroweak radiative correction introduced in Ref. [15]. Before the work of Refs. [7,8], only the terms involving c_1 in Eq.(6) were known and $C(m_e)$ was approximated by the expression

$$\frac{\alpha}{\pi}c_1\left[1 + \frac{2\alpha}{3\pi}\ln\frac{m_\mu}{m_e}\right].$$

In Eq.(26), $f(m_e^2/m_\mu^2)m_\mu^5/192\pi^3$ is a phase space factor and $3m_\mu^2/5M_W^2$ is the tree-level contribution of the W-propagator.

If terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$ are neglected in the radiative corrections, and Δr is approximated by $\Delta r^{(1)} - e^4 \mathrm{Re}\Pi_2^{(r)}(M_Z^2)$, where $\Delta r^{(1)}$ is the one-loop contribution to Δr, and $e^4\Pi_2^{(r)}(M_Z^2)$ is the two-loop contribution to the renormalized vacuum polarization function at $q^2 = M_Z^2$, it was shown in Ref. [16] that Eq.(25) contains all the one-loop corrections, as well as all two-loop contributions involving mass singularities (i.e. logarithms $\ln m$, where m is a generic light fermion mass). It also contains all the terms of $\mathcal{O}\left[(\alpha\ln(M_Z/m))^n\right]$. In fact, if so desired, all these logarithms can be absorbed by expressing $g^4/(1-\Delta r)^2$ in terms of the running coupling $\alpha(M_Z) = \alpha/(1-\Delta\alpha)$, where $\Delta\alpha = -\mathrm{Re}\Pi^{(r)}(M_Z^2)$. This result follows from the observation that, in $g^4/(1-\Delta r)^2$, such logarithms arise from the renormalization of α_0 in terms of α.

We now show that, if contributions of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$ are neglected in the radiative corrections, the factorization displayed in Eq.(25), involving the QED corrections of the Fermi theory and the electroweak factor $g^4/(1-\Delta r)^2$, is valid to all orders in perturbation theory. This applies not only to the corrections to the lifetime, but also to those affecting the electron spectrum, as well as the momentum dependence of the asymmmetry and the integrated asymmetry in the case of polarized muons. We present two arguments, one based on the effective field theory approach [17], the other involving a simple discussion of higher order Feynman diagrams.

If contribution of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$ are neglected in the radiative corrections and the tree-level term $3m_\mu^2/5M_W^2$ in Eq.(26) is for the moment disregarded, the effective field theory at the muon mass scale is the local V-A four-fermion Lagrangian density

$$\frac{G_F}{\sqrt{2}}\left[\overline{\Psi}_e\gamma^\mu(1-\gamma^5)\Psi_{\nu_e}\right]\left[\overline{\Psi}_{\nu_\mu}\gamma_\mu(1-\gamma^5)\Psi_\mu\right],$$

plus QED, plus QCD. Therefore, one can systematically evaluate the corrections to the spectrum, lifetime, and asymmetry in muon decay on the basis of this effective Lagrangian. This procedure results in the usual expressions involving G_F and the radiative corrections of the Fermi V-A theory [1]. As the latter are convergent to all orders in α [3], there is no need to cancel ultraviolet divergences and, therefore, in their expression, there is no reference to the high mass scale M_Z of the underlying theory. As mentioned in the Introduction, these corrections are known at the one-loop level in the case of the electron spectrum and asymmetry, and have now been evaluated at the two-loop level in the case of $1/\tau_\mu$. In particular one finds

$$\frac{1}{\tau_\mu} = \frac{G_F^2 m_\mu^5}{192\pi^3} f(m_e^2/m_\mu^2)\,[1 + C(m_e)]\,, \tag{28}$$

where $C(m_e)$ is the two-loop correction discussed in Sections 1 and 2. This leads to Eq.(25) provided we identify

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \frac{1 + \frac{3}{10}\frac{m_\mu^2}{M_W^2}}{(1 - \Delta r)}\ . \tag{29}$$

Eq.(29) has a very simple interpretation: it is the matching relation that expresses the coupling constant of the effective, low energy theory, in terms of the coupling constants and radiative corrections of the underlying theory. It is very important to note that $C(m_e)$ and $g^2/(1-\Delta r)$ are separately scale-independent quantities. The μ-independence of $C(m_e)$ follows from the fact that, when expressed in terms of physical parameters such as α, m_e, m_μ, it is convergent to all orders of perturbation theory [3]. The μ-independence of $g^2/(1-\Delta r)$ follows then from the fact that it can be expressed in terms of physical observables via Eqs.(28,29). As explained above, in the on-shell scheme of renormalization g^2 is defined in terms of e^2, M_W and M_Z, which are physical quantities and, therefore, μ-independent. It follows that Δr is μ-independent, an important property that has been verified at the one-loop level [15], and through terms of $\mathcal{O}(g^4 M_t^2/M_W^2)$ at the two-loop level [18].

The same conclusion, concerning the factorization of QED and electroweak corrections when terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$ are neglected, can be reached by a simple analysis of Feynman diagrams. Consider, for instance, a two-loop diagram involving a photon of virtual momentum k attached to external μ and/or e lines and a second loop that includes heavy particles. The relevant momenta in the QED correction of the Fermi theory are $|k^2| \lesssim m_\mu^2$. If we neglect terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$, we can set $k = 0$ in the heavy-loop integration and the two-loops factor out. We also note that when $|k^2|$ becomes a few times larger than m_μ^2, we can set $p_e = 0$, where p_e is the electron four-momentum. All reference to p_e is lost and we see that such domain of virtual momenta does not contribute to the corrections to the electron spectrum. Thus, the latter involves essentially the domain $|k^2| \lesssim m_\mu^2$, for which the factorization is valid. A second consideration, based on Feynman diagrams, refers to the factorization of QED corrections relative to the dominant electroweak corrections. These arise from the renormalization of the bare coupling constant $g_0^2 = e_0^2/s_0^2$. Consider, for simplicity, the one-loop photonic diagrams to μ-decay in the SM. Each diagram carries a factor e_0^2/s_0^2 associated with the virtual W interchange. The renormalization of e_0^2/s_0^2 in such diagrams leads, in higher orders, to dominant electroweak corrections that clearly factorize with respect to the QED corrections. The same arguments can be extended to higher orders.

The crucial Eq.(29) can be simplified by introducing a coupling constant

$$G_\mu = \frac{G_F}{1 + \frac{3m_\mu^2}{10M_W^2}},$$

so that Eqs.(28,29) become

$$\frac{1}{\tau_\mu} = \frac{G_\mu^2 m_\mu^5}{192\pi^3} f(m_e^2/m_\mu^2) \left[1 + \frac{3}{5}\frac{m_\mu^2}{M_W^2}\right] [1 + C(m_e)] \,, \tag{30}$$

$$\frac{G_\mu}{\sqrt{2}} = \frac{g^2}{8M_W^2}\frac{1}{(1-\Delta r)} \,. \tag{31}$$

These are, in fact, the expressions most commonly used in literature. The difference between G_μ and G_F is 0.5 ppm; it is completely negligible at present and it would only be of marginal interest if the experimental error in τ_μ is reduced by a factor 10. Nonetheless, this factor is frequently included in the theoretical expressions, as it is a tree-level contribution from the SM. Eq.(30) has the convenient feature that all the kinematical factors of the SM calculation are separated out.

The arguments carried above can be applied mutatis mutandi to other renormalization frameworks such as the $\overline{MS}$ scheme of Ref. [19], in which couplings in $g^2/(1-\Delta r)$ are identified with running $\overline{MS}$ parameters evaluated at M_Z, masses are still interpreted as pole masses, and Δr is modified accordingly.

The authors of Ref. [8] discuss a renormalization framework in which the couplings in $g^2/(1-\Delta r)$ are also identified with running parameters evaluated at M_Z. In particular, they claim that, in order to ensure the consistency of the renormalization scheme, the QED correction C of the local theory, when applied to the SM, must be evaluated via Eq.(1) with $\alpha(m_\mu) \to \alpha_e(M_Z)$ ($\alpha_e(M_Z)$ is the expression obtained from Eq.(3) by replacing $m_\mu \to M_Z$). This claim has a curious consequence: before c_2 was evaluated, the current experimental value of τ_μ led, via Eq.(28), to $G_F = 1.16639(2) \times 10^{-5}\ /GeV^2$. With the incorporation of the c_2 term in Eq.(1) one obtains $G_F = 1.16637(1) \times 10^{-5}\ /GeV^2$. However, according to the new claim, the correction C is to be evaluated differently when applied to the SM than in the Fermi theory, with the result that the relevant value is turned back to $G_F = 1.16639(1) \times 10^{-5}\ /GeV^2$!

In order to clarify this strange situation, we make the following observations:
i) As explained above, both the QED correction C and the electroweak amplitude $g^2/(1-\Delta r)$ are separately μ-independent quantities. One can certainly evaluate these corrections using running couplings. In this case, if the expressions are carried out to all orders, the μ-dependence of the couplings is cancelled by the μ-dependence of the radiative corrections. In actual calculations, due to the necessary truncation of the perturbative series, a μ-dependence emerges. However, there is no reason to require that the same scale be employed in $g^2/(1-\Delta r)$ and in $C(\mu)$. Indeed, the natural scale in the former is $\mu \approx M_Z$ and, in the latter, it is somewhere in the range $m_e < \mu < m_\mu$, with the optimization methods discussed in Section 2 favoring the lower values of μ. For instance, part of C are I.B. contributions. The photons here carry $k^2 = 0$ and it is clear that their natural coupling is $\alpha(m_e) = \alpha$, in sharp contrast with $\alpha_e(M_Z)$.
ii) Although the choice $\mu = M_Z$ for the QED correction is not natural, its implementation should be done by setting $\mu = M_Z$ in Eq.(5), rather than replacing $\alpha(m_\mu) \to \alpha_e(M_Z)$ in Eq.(1). Using Eq.(5) and $\alpha_e(M_Z)$ to evaluate $C(M_Z)$, we find that G_F would be decreased by only 0.6 ppm rather than increased by 20 ppm. In particular, we note that changing the scale of $\alpha(\mu)$ in Eq.(1), without modifying the second order coefficient according to Eq.(5), is inconsistent at the two-loop level.

In summary, our conclusion is that, subject to the neglect of terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$, one should apply the same QED corrections in the SM as in the Fermi theory. In particular, the value $G_F = 1.16637(1) \times 10^{-5}\ /GeV^2$, obtained in the Fermi theory, is the one that should be applied in the analysis of the SM.

IV. CONCLUSIONS

In Section 2, we have applied the traditional FAC, PMS and BLM optimization methods to the radiative corrections to the muon lifetime in the Fermi V-A theory. As the corrections are convergent, one expects on general grounds the natural mass scale to be in the range $m_e < \mu < m_\mu$. The analysis shows that the FAC and PMS approaches select a mass scale very close to m_e, while the BLM scheme leads to one very near the geometric average $\sqrt{m_e m_\mu}$ of the two masses. The FAC and PMS methods provide nearly identical estimations of the third order coefficient in the $\alpha(m_\mu)$ expansion, while the BLM estimation is of the same sign and differs only by 20% in magnitude. The three optimized expansions give very close results, with a maximum absolute difference of 5.4×10^{-8}. We use the maximal difference between the optimized and $\alpha(m_\mu)$ expansion, which amounts to be 2.6×10^{-7}, as an estimate of the theoretical error due to the truncation of the perturbative series. This translates into an error of 1.3×10^{-7} in the determination of G_F, a result very close to estimates obtained in Ref. [8] by different considerations. We also find that the expansion in powers of the conventional fine structure constant α, which in this case is essentially equivalent to the FAC expansion, contains a very small second-order coefficient. In fact, the second-order term is about 2900 times smaller than the first. Thus, it turns out that, by a curious cancellation of two-loop effects, the original one-loop calculation [1], expressed in terms of α, had an error of only 1.4×10^{-6}! Of course, this amusing historical fact could not be known before the work of Refs. [7,8] was carried out.

In Section 3, we return to the radiative corrections to muon decay in the SM. Using arguments based on the effective field theory approach, as well as considerations involving higher order Feynman diagrams, we show that, subject to the neglect of terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$, the overall answer factorizes into two separately scale-independent corrections: those of the Fermi V-A theory and the electroweak amplitude $g^2/(1-\Delta r)$. This important property applies to the corrections to the various observables, such as the electron spectrum, electron asymmetry in the case of polarized muons, and the muon lifetime. Therefore, if running couplings are employed, the scales may be chosen judiciously and independently in both factors, with $\mu \approx M_Z$ being the natural scale in $g^2/(1-\Delta r)$, and $m_e < \mu < m_\mu$ being the logical range in the corrections of the Fermi V-A theory. We reach the conclusion that, subject to the neglect of terms of $\mathcal{O}(\alpha m_\mu^2/M_W^2)$, one should apply the same QED corrections to muon decay in the SM as in the Fermi theory. In particular, at variance with a claim presented in Ref. [8], we find that the value $G_F = 1.16637(1) \times 10^{-5}/GeV^2$, currently obtained in the Fermi theory, is the one that should be applied in the analysis of the SM.

We conclude our discussion with the following comments:

i) Although the shifts discussed in Refs. [7,8] and this paper are numerically very small, it is clearly desiderable to evaluate fundamental parameters such as G_F, as accurately as possible.

ii) At present, the Fermi V-A theory is generally viewed, not as independent, but rather as an effective low-energy theory derived from the SM. The arguments presented in this paper make clear that the same constant G_F that is precisely determined in the low-energy theory is, to a high degree of accuracy, the relevant paramenter in the analysis of the more fundamental, underlying gauge theory.

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