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Complete 2-loop Quantum Electrodynamic Contributions to the Muon Lifetime in the Fermi Model

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The complete 2-loop quantum electrodynamic corrections to the muon lifetime are calculated in the Fermi theory. The exact result for the effects of virtual and real photons, virtual electrons, muons as well as e^+e^- pair creation is

$$\Delta\Gamma^{(2)}_{\rm QED} = \Gamma_0\left(\frac{\alpha}{\pi}\right)^2\left(\frac{156815}{5184} - \frac{1036}{27}\zeta(2) - \frac{895}{36}\zeta(3) + \frac{67}{8}\zeta(4) + 53\zeta(2)\ln(2)\right) = \Gamma_0\left(\frac{\alpha}{\pi}\right)^2 6.743$$

where Γ_0 is the tree-level width. The theoretical error in the value of the Fermi coupling constant, G_F, is now rendered negligible compared to the experimental uncertainty coming from the measurement of the muon lifetime. The overall error in G_F is then roughly halved giving

$$G_F = (1.16637 \pm 0.00001)\times 10^{-5}\ {\rm GeV}^{-2}.$$

13.35.Bv, 13.40.Ks, 14.60.Ef, 12.15.Lk, 12.20.Ds

I. INTRODUCTION

The Fermi coupling constant, G_F, plays a key rôle in precision tests of the Standard Model of electroweak interactions. Along with the electromagnetic coupling constant, α, and the Z boson mass, M_Z, it is one of the best measured quantities of electroweak physics and as such is used as input in all higher-order calculations. G_F is one of the few quantities that is sensitive to physics at very high energy scales and is intimately related to the ρ parameter [1]. It was the value of G_F that provided some of the strongest constraints on the mass of the top quark before it was directly observed.

G_F is extracted from measurements of the muon lifetime, $\tau_\mu \equiv \Gamma_\mu^{-1}$, which is a purely leptonic process and therefore very clean both experimentally and theoretically. Its quoted error is $\delta G_F/G_F = 1.7\times10^{-5}$ of which 0.9×10^{-5} is experimental and 1.5×10^{-5} is theoretical; the latter being an estimate of the size of the 2-loop corrections. Experiments are under consideration at Brookhaven National Laboratory, the Paul Scherrer Institute, and the Rutherford-Appleton Laboratory that could lead to a reduction in the experimental error on the τ_μ of a factor of 10 or more.

The radiative corrections to muon decay in the full Standard Model naturally factorize into two pieces [2], one of which, to a very high degree accuracy, is just the quantum electrodynamic (QED) radiative corrections in the Fermi theory. The other piece is left free of infrared singular contributions. It contains purely weak corrections that can be absorbed into G_F which then possesses an enriched sample of weak sector physics. Such a separation between QED and weak corrections is not generally possible for charged current processes.

The 1-loop QED contributions to the muon lifetime were first calculated over 40 years ago by Kinoshita and Sirlin [3] and by Berman [4]. It is known [5] that the Fermi theory in the presence of QED is finite to leading order in G_F and to all orders in the electromagnetic coupling constant, α. This remarkable fact means that G_F can be defined in a physically unambiguous manner at least up to the point where finite W propagator effects begin to appear.

In this article the 2-loop QED radiative corrections to muon lifetime are calculated in the Fermi theory. The result is used to extract an improved value for G_F in which the error is entirely due to the experimental uncertainty.

II. THE FERMI COUPLING CONSTANT

The Fermi theory Lagrangian, relevant for the calculation of the muon lifetime is

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$$\mathcal{L}_F = \mathcal{L}^0_{\rm QED} + \mathcal{L}^0_{\rm QCD} + \mathcal{L}_W \qquad (1)$$

Here $\mathcal{L}_W$ is the Fermi contact interaction

$$\mathcal{L}_W = -2\sqrt{2} G_F \left[\bar{\psi}^0_{\nu_\mu} \gamma_\lambda \gamma_L \psi^0_\mu\right] . \left[\bar{\psi}^0_e \gamma_\lambda \gamma_L \psi^0_{\nu_e}\right] \qquad (2)$$

in which ψ_μ, ψ_e, ψ_{ν_μ} and ψ_{ν_e} are the wave functions for the muon, the electron and their associated neutrinos respectively. The Euclidean metric in which time-like momenta squared are negative is used. $\mathcal{L}^0_{\rm QCD}$ is the bare Quantum Chromodynamic (QCD) Lagrangian responsible for the strong interactions and $\mathcal{L}^0_{\rm QED}$ is the usual bare Lagrangian of QED,

$$\mathcal{L}^0_{\rm QED} = -\sum_f \bar{\psi}^0_f (i\not{\partial} + m_f) \psi^0_f - \frac{1}{4} \left(\partial_\rho A^0_\sigma - \partial_\sigma A^0_\rho\right)^2 + i e^0 \sum_f Q_f \bar{\psi}^0_f \gamma_\rho \psi^0_f A^0_\rho . \qquad (3)$$

The sum is over all fermion species, f, with mass, m_f, and electric charge, Q_f. A_ρ is the photon field and $\gamma_L = \frac{1}{2}(1+\gamma_5)$ denotes the usual Dirac left-hand projection operator. The superscript ^{0} indicates bare, as opposed to renormalized, quantities. For the present purposes G_F goes unrenormalized. Throughout this article dimensional regularization [6] is used for the ultraviolet (UV) divergences. The appearance of infrared (IR) divergences is largely avoided by the methods employed here.

The formula obtained for τ_μ by means of the $\mathcal{L}_{\rm F}$ is finite to leading order in G_F and all orders in the renormalized electromagnetic coupling constant, $\alpha_r = e_r^2/(4\pi)$ [5]. This follows from the fact that under a Fierz rearrangement that interchanges the wavefunctions $\bar{\psi}_e$ and $\bar{\psi}_{\nu_\mu}$ in $\mathcal{L}_W$ the currents remain purely left-handed vector currents. This is in sharp contrast to the case of neutron decay in which scalar and pseudoscalar terms are generated and for which the following arguments break down. The radiative corrections in that case are not finite. Considering the vector part, $\bar{\psi}_e \gamma_\mu \psi_\mu$, of this effective μ-e current, one sees that after fermion mass renormalization is performed the remaining divergences are independent of the masses and thus cancel, as for the case of pure QED. The QED corrections to the axial vector part may be shown to be finite by noting that the transformations $\psi_e \to \gamma_5 \psi_e$ and $m_e \to -m_e$ leave $\mathcal{L}_{\rm QED}$ and $\mathcal{L}_{\rm QCD}$ invariant but exchanges $\bar{\psi}_e \gamma_\lambda \psi_\mu \leftrightarrow \bar{\psi}_e \gamma_\lambda \gamma_5 \psi_\mu$. Thus the radiative corrections to the axial-vector part of the current are equal to those of the vector part in the limit of $m_e = 0$. In practice then the only radiative corrections to the vector pieces in $\mathcal{L}_W$ need to be calculated which avoids entirely the problems associated with γ_5 in dimensional regularization.

To lowest order in G_F the expression for the muon lifetime calculated from $\mathcal{L}_F$ takes the general form

$$\frac{1}{\tau_\mu} \equiv \Gamma_\mu = \Gamma_0 (1 + \Delta q). \qquad (4)$$

where

$$\Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3}$$

and Δq encapsulates the higher order QED and QCD corrections generated by $\mathcal{L}_F$ and can be expressed as a power series expansion in the renormalized electromagnetic coupling constant, α_r,

$$\Delta q = \sum_{i=0}^{\infty} \Delta q^{(i)} \qquad (5)$$

in which the index i gives the power of α_r that appears in $\Delta q^{(i)}$.

Assuming that the electron neutrino and muon neutrino are massless it can be shown that

$$\Delta q^{(0)} = -8x - 12x^2 \ln x + 8x^3 - x^4, \quad x = \frac{m_e^2}{m_\mu^2}, \qquad (6)$$

that comes from phase space integrations.

The $\mathcal{O}(\alpha)$ corrections in Δq, first obtained by Kinoshita and Sirlin [3] and by Berman [4], are

$$\Delta q^{(1)} = \left(\frac{\alpha_r}{\pi}\right) \left(\frac{25}{8} - 3\zeta(2)\right) + \mathcal{O}\left(\alpha_r \frac{m_e^2}{m_\mu^2} \ln \frac{m_e^2}{m_\mu^2}\right). \qquad (7)$$

where ζ is the Riemann zeta function and $\zeta(2) = \pi^2/6$. An exact expression for the full electron mass dependence in $\Delta q^{(1)}$ has been given by Nir [7].

Recently the hadronic contributions to $\Delta q^{(2)}$ were computed using dispersion relations along with contributions from muon and tau loops [8]. Their effect was shown to be small relative to the present experimental error. They become relevant for the next generation of muon lifetime experiments but the hadronic uncertainty is still well under control.

The Kinoshita-Lee-Nauenberg [9] theorem guarantees that Δq is free from singularities as $m_e \to 0$, other than those that can are absorbed into α_r. It may be shown [10] that all large logarithms of the form $\alpha^i \ln^{i-1}(m_\mu^2/m_e^2)$ for all $i > 0$ and those of $\alpha^3 \ln(m_\mu^2/m_e^2)$ can be accounted for, in a manner consistent both with the calculation of Ref. [8] and the perturbative results presented here, by setting

$$\alpha_r \longrightarrow \alpha_e(m_\mu) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\ln\frac{m_\mu^2}{m_e^2}} + \frac{\alpha^3}{4\pi^2}\ln\frac{m_\mu^2}{m_e^2} \tag{8}$$

where α is the experimentally-measured quantity, $\alpha = 1/137.0359895(61)$ [11]. The contribution to the muon lifetime from the $\mathcal{O}(\alpha^2)$ logarithmic term coincides with the result obtained in Ref. [12]. The logarithms of $\mathcal{O}(\alpha^3)$ were first obtained by Jost and Luttinger [13]. When evaluated Eq.(8) yields $\alpha_e(m_\mu) = 1/135.90 = 0.0073582$. In the $\overline{\mathrm{MS}}$ renormalization scheme with 't Hooft mass, $\mu = m_\mu$, Eq.(8) correctly includes non-logarithmic terms up to $\mathcal{O}(\alpha^2)$ but those of $\mathcal{O}(\alpha^3)$ have been dropped.

III. 2-LOOP CORRECTIONS

A. Photonic Corrections

The calculation of the 2-loop QED corrections to the muon lifetime involves the sum of the cross-sections $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma\gamma$ and $\mu^- \to \nu_\mu e^- \bar{\nu}_e e^+ e^-$ with up to two virtual photons. Individual diagrams are IR divergent and in some cases require integration over a 5-body phase space. The problem of cancelling these IR singularities can be avoided entirely if the QED corrections are obtained as the imaginary part of 4-loop propagator type Feynman diagrams by means of the optical theorem. Some of these diagrams are shown in Fig.1. The heavy lines represent muons which are the only particles taken to have non-zero mass. The 4-fermion vertex used is the vector part of the usual one from the Fermi theory. Inspection of the diagrams shows that the cuts generating imaginary parts produce all of the Feynman diagrams contributing to muon decay. Extra diagrams do appear in which the cut goes through a muon line but such diagrams vanish kinematically because the external muon is on its mass shell.

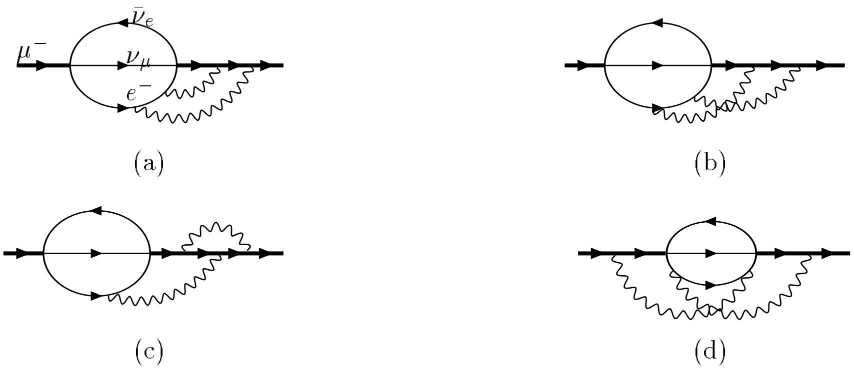

FIG. 1. Examples of diagrams whose cuts give contributions to $\mu^- \to \nu_\mu e^- \bar{\nu}_e$, $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma$ or $\mu^- \to \nu_\mu e^- \bar{\nu}_e \gamma\gamma$.

The imaginary parts of the necessary 4-loop propagator type diagrams were calculated as follows. Recursion relations [14] obtained by integration-by-parts were first applied to reduce all dimensionally regularized integrals to a small set of relatively simple integrals. These primitive integrals were chosen to be free from specific IR divergences that occur on-shell. The well-behaved primitive integrals were then calculated by taking the external muon momentum, q, off mass shell to obtain expressions as power series in $x = -q^2/m_\mu^2$ and logarithms of x using well-established large mass expansion techniques along the lines of Ref. [15]. This series serves as a convenient representation as its coefficients involve simpler integrals. Many terms in the large mass expansion can be discarded since they do not contribute to the imaginary part. What remains of the coefficients in the expansion can be evaluated in closed form in terms of polygamma functions and certain classes of multiple nested sums [16]. Then, the on-shell limit, $x = 1$, is taken and the infinite sum over the coefficients of x^k is performed. In this process the exact expressions collapse into known constants such as the Riemann zeta function of integer arguments, $\zeta(k)$, and polylogarithms, $\mathrm{Li}_k(1/2)$. Details of the procedures followed will be described elsewhere [10].

Fermion mass renormalization is performed in the on-shell scheme[1] that generates derivatives of fermion self-energies for the external leg corrections. All diagrams were calculated in a general covariant gauge for the photon field and exact cancellation in the final result of the dependence on the gauge parameter was demonstrated.

The result for just the photonic diagrams is

$$\Delta\Gamma^{(2)}_{\gamma\gamma} = \Gamma_0 \left(\frac{\alpha_e(m_\mu)}{\pi}\right)^2 \left(\frac{11047}{2592} - \frac{1030}{27}\zeta(2) - \frac{223}{36}\zeta(3) + \frac{67}{8}\zeta(4) + 53\zeta(2)\ln(2)\right) \tag{9a}$$

$$= \Gamma_0 \left(\frac{\alpha_e(m_\mu)}{\pi}\right)^2 3.55877. \tag{9b}$$

B. Electron Loop Corrections

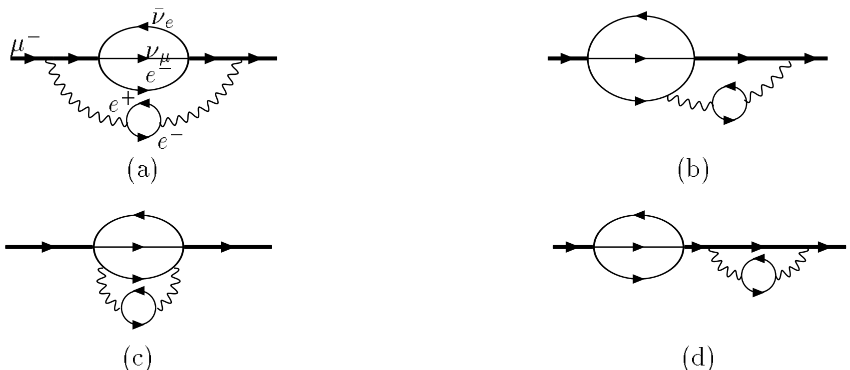

FIG. 2. Diagrams containing an electron loop whose cuts give contributions to muon decay, $\mu^- \to \nu_\mu e^- \bar\nu_e$, $\mu^- \to \nu_\mu e^- \bar\nu_e \gamma$ or $\mu^- \to \nu_\mu e^- \bar\nu_e e^+ e^-$.

The contribution of electron loops to the muon lifetime differs from those of other fermions in that they must be combined with diagrams with an additional e^+e^- pair in the final state in order to produce an IR finite result, however the procedure described above may be applied here as well. The electron loop diagrams are shown in Fig.2. To Fig.2d must be added a diagram containing a muon mass counterterm, δm_μ, on the external leg. Furthermore, diagrams, in which the electron loop is replaced by the photon self-energy counterterm, must be included to produce a UV finite result. This counterterm contribution is proportional to $\Delta q^{(1)}$ and depends on the particular renormalization scheme that has been chosen. The overall result in the $\overline{\mathrm{MS}}$ renormalization scheme with 't Hooft mass $\mu = m_\mu$ consistent with Eq.(8) is

$$\Delta\Gamma^{(2)}_{\mathrm{elec}} = -\Gamma_0 \left(\frac{\alpha_e(m_\mu)}{\pi}\right)^2 \left(\frac{1009}{228} - \frac{77}{36}\zeta(2) - \frac{8}{3}\zeta(3)\right) \tag{10a}$$

$$= \Gamma_0 \left(\frac{\alpha_e(m_\mu)}{\pi}\right)^2 3.22034 \tag{10b}$$

[1]That is to say that the renormalized mass of a stable fermion is set equal to its physical or pole mass

which is about two orders of magnitude greater than that of either muon loops or hadrons. The value obtained in Eq.(10b) is consistent with a numerical study presented in Ref. [17] in the context of semi-leptonic decays of heavy quarks.

The same methods used to calculate the contribution from electron loops can be applied to muon loops. Agreement was found with the result of Ref. [8].

IV. CONCLUSIONS

The photonic corrections of section III A can be combined with those of the electron loops and e^+e^- pair production of section III B, and adding the exact result for muon loops of Ref. [8] gives

$$\Delta\Gamma^{(2)}_{\rm QED} = \Gamma_0 \left(\frac{\alpha_r}{\pi}\right)^2 \left(\frac{156815}{5184} - \frac{1036}{27}\zeta(2) - \frac{895}{36}\zeta(3) + \frac{67}{8}\zeta(4) + 53\zeta(2)\ln(2)\right) \tag{11a}$$

$$= \Gamma_0 \left(\frac{\alpha_r}{\pi}\right)^2 6.743 \tag{11b}$$

with $\alpha_r = \alpha_e(m_\mu)$=1/135.90. The resulting expression contains all corrections of $\mathcal{O}(\alpha^2)$, $\mathcal{O}\left(\alpha^3 \ln(m_e^2/m_\mu^2)\right)$ and $\mathcal{O}\left(\alpha^i \ln^{i-1}(m_e^2/m_\mu^2)\right)$ for all $i \geq 2$. Adding the hadronic and tau loop contributions of Ref. [8] one obtains

$$\Delta\Gamma^{(2)} = \Gamma_0 \left(\frac{\alpha_r}{\pi}\right)^2 (6.700 \pm 0.002). \tag{12}$$

where the error is a conservative estimate of the hadronic uncertainty. Using the current best value for $\tau_\mu = (2.19703 \pm 0.00004)\,\mu$s [11] yields

$$G_F = (1.16637 \pm 0.00001) \times 10^{-5}\,\mathrm{GeV}^{-2}. \tag{13}$$

That represents a reduction in the overall error on G_F of about a factor of 2 and a downward shift in the central value of twice the experimental uncertainty. G_F is now known to 9 ppm. The next generation of measurements of the muon lifetime are expected to reduce this by at least a further factor of 10.

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