Nuclear muon capture on the proton and $^3$He within the standard model and beyond

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Abstract

Nuclear muon capture on the proton and $^3$He is considered both within and beyond the standard model in terms as general as is possible. Explicit and precise analytic expressions for all possible observables are given, assuming only a Dirac neutrino in the limit of vanishing mass. These results allow both for precision tests of the standard model and new physics, as well as for the assessment of the potential physics reach of experiments designed to measure specific observables. Using these expressions, stringent constraints can already be inferred from a recent precision measurement of the statistical capture rate on $^3$He. Likewise, similar constraints should follow the completion of a precision measurement in progress of the singlet capture rate on the proton. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Over the last fifteen years, muon physics has regained its rightful place in particle physics, ranging from intermediate energies well into the high energy frontier in the foreseeable future with the advent of muon colliders. Given the availability of intense muon beams at different laboratories, as well as new and much efficient experimental and detector techniques, intermediate energy muon physics has moved into the realm of precision studies of the standard model, with the hope of possibly unravelling some tell-tale sign for the physics which must lie beyond it in ways complementary to present day physics.
high-energy experiments at the colliders. Such studies include both purely leptonic as well as semileptonic electroweak processes, the much studied field of nuclear muon capture (for reviews, see [1–3]) belonging to the latter class.

When the aim of such electroweak processes in nuclei is to study particle physics issues, the uncertainties inherent to nuclear structure modeling have to be disposed of to the largest extent possible, leaving essentially only the lightest of nuclei available, beginning with the proton and next the stable 3-nucleon bound state, $^3$He, since indeed the quantum wave functions for the latter system are by now very well understood [4,5]. An additional bonus in the case of these two systems is that they both provide the unique occurrence of a (mother–daughter) pair of nuclei which define a spin 1/2 isospin doublet, namely the proton and neutron, and $^3$He and $^3$H, respectively. Hence in the limit of exact isospin symmetry, some of the various phenomenological nuclear form factors which parameterize matrix elements of these states are related to one another, while for spin 1/2 states, the numbers of these form factors remains small. This is how through CVC, the experimental determination of electromagnetic form factors from electron scattering may be translated into knowledge of the related nuclear form factors for the charged electroweak vector current. In other words, hiding our ignorance of the microscopic dynamics at the quark level into a phenomenological parameterization in terms of form factors, it remains possible to consider predictions of observables which do not require models for nuclear structure. In the field of nuclear muon capture, this description corresponds to the so-called “elementary particle model” approach to nuclear electroweak processes [6], which will be used in this paper.

One of the main motivations for studying nuclear muon capture, especially in light nuclei, has always been to measure the induced pseudoscalar nucleon form factor $g_P(q^2)$, certainly the least well-known of all nonvanishing nucleon form factors with a combined uncertainty which has stood at 22% for the last twenty years [7]. The urgency of this specific issue has recently become more pressing, mainly for two reasons. On the one hand, based on the fundamental chiral symmetries which, even though dynamically broken, survive nonlinearly in the nonperturbative low energy regime of the theory for the strong interactions among quarks, namely quantum chromodynamics, definite predictions for $g_P$ with a precision of a few percents or better have been achieved [8–11]. The experimental confirmation of the expected value is thus a crucial low-energy test of our basic concepts for the theory of the strong interactions. On the other hand, a new 8% precise measurement of $g_P$ has been completed in the intervening years, using the rare process of radiative muon capture on hydrogen [12,13]. The unsettling fact is that the value obtained differs significantly (by a 4.2 $\sigma$ deviation) from the expected value. No theoretical explanation for this discrepancy having been found so far (see for example Ref. [11] for references), the issue thus remains open and this difficult experiment shrouded with some feeling of uncertainty. Indeed, the general understanding of the dynamical breaking of the

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3 In heavier nuclei, the same axial electroweak probe enables to address the issue of hadronic coupling and mass renormalization in the nuclear medium.
chiral symmetries of QCD has so far never been found lacking in essentially any other low-energy process.

During the same period, another precision measurement of nuclear muon capture on $^3\text{He}$ has also been completed [14]. In spite of its remarkable precision of 0.4% for the statistical capture rate at $\lambda_\text{stat}^{\exp} = 1496 \pm 4 \text{ s}^{-1}$, whose value is found to be in perfect agreement with the theoretical prediction $\lambda_\text{theor}^{\text{stat}} = 1496 \pm 21 \text{ s}^{-1}$ [4,5], the ensuing value for $g_P$, though at exactly the expected level, is still precise to only 19% [15]. One should add however that in terms of the corresponding pseudoscalar nuclear form factor $F_P(q^2)$ for $^3\text{He}$, the same experimental result translates [14,16–19] into a 13% precise test of the prediction based on PCAC, the symmetry which together with CVC has historically been the precursor for the chiral symmetries of QCD. Even though these results are still a long way off the precision reached by the latest theoretical analyses [8–11], they stand as a clear confirmation of the basic concepts involved in the chiral symmetry aspects of the problem.

Independently of this specific situation with respect to the value for $g_P$, if one is willing to use the theoretically expected number, the latest ordinary muon capture experiment on $^3\text{He}$ has reached such a level of precision that other tests of the standard model (SM) for the electroweak interactions become possible [17–19], some of which prove to be quite stringent for possible new physics beyond the SM. Moreover, given the above issues surrounding the value of $g_P$, as well as the precision of its theoretical prediction, a new effort has been launched [16,20] in order to measure the singlet rate of ordinary muon capture on hydrogen in a gas target (to avoid the complications due to molecular binding effects), hopefully to a precision of 0.5% to 1%. Here again, beyond the initial aim towards the value of $g_P$, this level of precision should also enable tests of the SM. However, in order to infer a value for $g_P$ from any experiment, analytic expressions for observables whose numerical evaluation is to the required standard of precision should be available. This is the purpose of the present work, much in continuation of that of Refs. [4,5].

This paper considers the ordinary nuclear muon capture process on a spin 1/2 isospin doublet in terms as general as is possible. Not only are all possible effects existing within the SM included in the present analysis, but any possible contributions which may appear beyond the SM due to new interactions are accounted for as well through an effective four-fermi quark–lepton interaction which describes all possible scalar, pseudoscalar, vector, axial and tensor couplings. Indeed, given the momentum transfer involved, much less than any of the mass scales characterizing such possible interactions, an effective four-Fermi interaction at the quark–lepton level is perfectly justified.

Previous analyses over the years, beginning with Ref. [21] pointing out the existence of the hyperfine effect in the capture rates because of maximal $(V - A)$ parity violation, have all only included vector and axial interactions through the usual $(V - A)$ charged electroweak coupling, and only some of these works have considered the possibility of

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4 The principle of the experiment lies in the comparison between the muon disappearance rates, through the usual electron decay mode, for both positive and negative muons, only the latter being subject to nuclear capture. Since this experiment does not measure the neutron recoil energy distribution, it has no handle on the neutrino energy spectrum nor on its mass.

5 The same effect arises of course also for a pure $(V + A)$ coupling.
so-called second-class currents. A handful of observables have been computed, and then not always in analytic form \cite{4–6,11,15,22–26}. In contradistinction, the present work does not involve any non relativistic expansion and allows for all possible contributions, including CP- or T-odd effects through complex form factors and coupling coefficients in the effective interaction. The only approximations made are, on the one hand, that the neutrino mass is taken to vanish and that the muon leptonic flavour is conserved, and on the other hand, that nuclear recoil effects associated to the scalar, pseudoscalar and tensor interactions beyond the SM are also ignored, since the latter couplings are necessarily small on their own and would in turn multiply small nuclear recoil contributions. Moreover, all possible observables become available through our results, hopefully making it easier to assess their sensitivity to whatever parameter or physical input a specific experiment using ordinary nuclear muon capture on these nuclei would wish to address. This point will explicitly be illustrated with some of the obtained observables, in the context of the two experiments mentioned previously.

Most of these observables require correlation measurements using polarized states for the initial muonic atom, while another subset requires to measure also the polarization states of the outgoing nucleus and/or neutrino. These are certainly experimental challenges bordering on the impossible for certain of these polarization observables, but some experimentalists take up the task. For example, there exists a first-generation experiment \cite{27} which measures the vector analyzing power of the outgoing triton in polarized muon capture on \(^3\)He. Even though the preliminary results are yet not precise enough to be of use in a theoretical analysis, they certainly demonstrate that this specific difficult challenge can be met.

More specifically, the physics reason for these difficulties is that during its atomic cascade down to the muonic atom ground state, the muon, even if initially polarized, suffers depolarization effects to a great extent, leaving over only a small fraction of its initial polarization \cite{1–3,28–30}. Once in the ground state, the degree of polarization may be increased again by external means \cite{27,31}, but not to any large degree and less than theoretically anticipated \cite{32}. In fact, both the initial nucleus and muon need to be polarized in order to end up with a muonic atom ground state polarized to any degree \cite{33}. Hence, because of these atomic physics issues, even though the expressions for all observables are now available in analytic form, any experiment which consists not only in a rate measurement is in essence extremely difficult to perform when aiming towards the demands of great precision.  

The outline of the paper is as follows. In the next section, the general parameterization of the capture amplitude is described, and the ensuing expressions for observables explicitly

\footnote{In this respect, the recent proposal of Ref. \cite{34} appears to be totally unrealistic, the more so since some of the possible contributions which have not been included in that analysis, such as recoil order effects, could also lead to contributions to the massive neutrino polarization states. Such effects are all accounted for in our analysis in the limit of a massless neutrino, including the possibility of \(T\)-violating couplings and currents other than \(V\) and \(A\). Note also that due to helicity constraints, transverse contributions to a massive neutrino polarization state produced in muon capture through only \(V\) and \(A\) couplings are necessarily suppressed by the ratio of the neutrino mass to its energy, and are thus unobservable.}
given. In Section 3, the situation for $^3$He is then considered in detail, first within the SM, and then beyond the SM, each time by using the experimental result of Ref. [14] for the statistical capture rate and also by illustrating the potential physics reach of some other final state distribution which does not entail a correlation observable except for an initially polarized muonic atom. In Section 4, the same considerations are applied again to the case of capture on the proton, based on the optimistic aim of a 0.5% precision in the result for the singlet capture rate [16,20]. Section 6 then translates some of the limits for physics beyond the SM obtained from the previous considerations, into limits for parameters of some specific models for such physics. The conclusions end our discussion, while further information relevant to the analysis is provided in three separate appendices.

2. Muon capture observables

2.1. Kinematics

Our notations for kinematics are as follows. Let $m_\mu$, $M_1$ and $M_2$ be the masses of the muon, and of the initial and final nuclei, respectively, with $M = (M_1 + M_2)/2$ the mean value of the latter two. As mentioned previously, the capture process is considered in the limit of a vanishing neutrino mass and no leptonic flavour mixing. Initially, the muon and nucleus form a muonic atomic bound state at rest, whose total rest-mass is denoted $p_s$, which differs from $m_\mu C M_1$ by the binding energy $Z$, $Z$ being the reduced mass of the bound state (expressions are given in units such that $c = 1$ and $\hbar = 1$ throughout).

Let then $\vec{p}$ be the momentum of the outgoing nucleus ($-\vec{p}$ is thus that of the outgoing neutrino), $v$ the energy of that nucleus and $\omega$ the energy of the neutrino, such that we have

$$v = |\vec{p}|, \quad \sqrt{s} = v + \omega, \quad v = \frac{s - M_2^2}{2\sqrt{s}}, \quad \omega = \frac{s + M_2^2}{2\sqrt{s}}. \quad (1)$$

With respect to polarization states, let $\hat{s}_1$ denote the normalized polarization vector of the initial spin 1/2 nucleus, as measured in its rest-frame, and $\hat{s}_2$ that of the final spin 1/2 nucleus also measured in its rest-frame. Similarly, $\hat{\mu}$ denotes the muon normalized polarization vector in its rest-frame, while $\hat{c}_\mu$ is the massless Dirac neutrino helicity.

Finally, the capture distribution is given by the expression,

$$\frac{d\Gamma_\lambda}{d\Omega_\vec{p}} = \frac{2|\psi_\lambda(0)|^2 C}{64\pi^3 m_\mu M_1 \sqrt{s}} |\mathcal{M}_\lambda|^2. \quad (2)$$

Here, $d\Omega_\vec{p}$ is the element of solid angle associated to the outgoing nucleus of momentum $\vec{p}$, $\mathcal{M}_\lambda$ is the capture amplitude associated to a neutrino of helicity $\lambda$, $\psi_\lambda(0)$ is the 1S state

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$^7$ In practice, there is no difference between a massless Dirac or Majorana neutrino in the case of purely $(V - A)$ or $(V + A)$ interactions, but the distinction becomes relevant as soon as other interactions are turned on, as done in this paper. By this assumption of a massless Dirac neutrino, we exclude the possibility of interference contributions between processes in which either the neutrino spinor field $\nu(x)$ or its charge conjugate $\bar{\nu}(x)$ would couple in the amplitudes to the same quarks and leptons rather than their antiparticles, namely it is assumed that the muon leptonic flavour is conserved in the massless neutrino limit.
muonic atom Coulomb wave function measured at the origin, and $C$ is a reduction factor which accounts for the effects of the nuclei finite size through the overlap of the muon and neutrino wave functions with the different nuclear electric and electroweak charge distributions of finite spatial extent. The detailed evaluation of this reduction factor $C$ is discussed in Appendix A both for $^3\text{He}$ and for the proton.

2.2. The amplitude

The parameterization for the effective interaction associated to muon capture, at the level of the $u$ and $d$ quarks, is taken to be

$$4\frac{g^2}{8M^2}V_{ud}\sum_{n_1, n_2=0, \pm} \left[ (\bar{\psi}_{n_1}^V)^* \bar{\psi}_{n_2} \sigma_{\mu\nu} P_{n_1} \gamma_{\mu} P_{n_2} u + (\bar{\psi}_{n_1}^S)^* \bar{\psi}_{n_2} \gamma_{\mu} P_{n_1} \mu \bar{d} P_{n_2} d \right] + \frac{1}{2}(\bar{\psi}_{n_1}^T)^* \bar{\psi}_{n_2} \sigma_{\mu\nu} P_{n_1} \mu \bar{d} \sigma_{\mu\nu} P_{n_2} d - u.$$  (3)

Here, $P_{\pm} = (1 \pm \gamma_5)/2$ are the chirality projectors, and $g$, $M$ and $V_{ud}$ are arbitrary real parameters which in the limit of the SM reduce to those of that model, namely $M = M_W$, $g^2/8M^2 = G_F/\sqrt{2}$ and $V_{ud} = \cos\theta_c$ being the Cabibbo–Kobayashi–Maskawa (CKM) quark flavour-mixing matrix element, in which case all coefficients $h_{\pm=\pm}^{V,T}$ also vanish except for $h_{-}^{V} = 1$. Finally, the coefficients $h_{n_1/n_2}^{V,S,T}$ are arbitrary complex coefficients associated to vector, scalar and tensor interactions, with $n_1$ (respectively, $n_2$) being the muon (respectively, $d$-quark) chirality, equal to the neutrino (respectively, $u$-quark) one for vector interactions and opposite to it for scalar and tensor interactions (the "*" symbol denotes complex conjugation throughout). Finally, without loss of generality, one may set $h_{T+} = 0 = h_{T-}$, because of the identity $\sigma^{\mu\nu} = i\epsilon^{\mu\nu\rho\sigma} \sigma_{\rho\sigma}/2$ which implies that the terms multiplied by $h_{T+}^{V,T}$ and $h_{T-}^{V,T}$ simply vanish identically.

This choice of parameterization is inspired by the one used (in the charge exchange form) for muon decay in terms of coefficients $g_{S,V,T}^{\pm\pm}$, with the first (respectively, second) lower index being the chirality of the electron (respectively, muon) [35]. Note that a similar effective four-Fermi interaction may be given for $\beta$-decay in terms of coefficients $f_{S,V,T}^{\pm\pm}$ with the electron then playing the role of the muon in (3).

To express the amplitude for nuclear muon capture in terms of the above parameterization, one also requires the hadronic matrix elements of the relevant quark operators in terms of the nuclear bound states. For this purpose in the case of a spin 1/2 isodoublet, let us introduce the spinor $\psi_1$ for the Dirac field of the spin 1/2 nucleus on which the muon is captured, while $\psi_2$ is the Dirac spinor for the final spin 1/2 nucleus. In momentum space, with $q^\mu = p_2^\mu - p_1^\mu$ being the momentum transfer of the process and $p_2^\mu$ (respectively, $p_1^\mu$) being the momentum of the final (respectively, initial) nucleus, we have the following parameterization in terms of $q^2$-dependent form factors:

$$\langle 2|\bar{\psi}_2 \gamma_{\mu} u|1 \rangle = \bar{\psi}_2 F_{V} \gamma_{\mu} + i F_{M} \sigma_{\mu\nu} \frac{q^\nu}{2M} + F_{S} \frac{q^\mu}{2M} \psi_1,$$  (4)

$$\langle 2|\bar{\psi}_2 \gamma_{\mu} \gamma_5 u|1 \rangle = \bar{\psi}_2 F_{A} \gamma_{\mu} \gamma_5 + F_{P} \gamma_{\mu} \frac{q^\nu}{m_\mu} + i F_{T} \sigma_{\mu\nu} \gamma_5 \gamma_5 \frac{q^\nu}{2M} \psi_1.$$  (5)
where $|1\rangle$ and $|2\rangle$ denote the initial and final quantum nuclear states, respectively.

This notation for form factors is the one used for the $^3\text{He}-^3\text{H}$ case, while it is more conventional in the proton–neutron case to denote the form factors for the vector and axial quark currents as $g_V, g_M, g_S, g_A, g_P$ and $g_T$, respectively for $F_V, F_M, F_S, F_A, F_P$ and $F_T$. We shall thus follow that convention in the case of the proton, and keep nevertheless the notation $G_S, G_P$ and $G_T$ for the form factors associated to the scalar, pseudoscalar and tensor quark operators even in the case of the proton. However, all expressions for observables in this section will be given in terms of the $F_{V,M,S,A,P,T}$ form factors. Note also that the contribution of the induced pseudoscalar form factor $F_P$ has conventionally been normalized to the muon mass, while those of all other recoil order form factors are normalized with respect to twice the mean nuclear mass $2M$.

In the case of $T$-invariant interactions, both the effective coupling coefficients $h_{S,V,T}$ as well as the nuclear form factors are all real quantities under complex conjugation. However in the present analysis, this restriction is not imposed, and all these parameters are assumed to take a priori complex values.

For the vector and axial current quark operators, one distinguishes first- and second-class form factors, $F_V, F_M, F_A$ and $F_P$ in the first case, and $F_S, F_T$ in the second case. On the basis of CVC, the values for the electroweak vector and induced magnetic form factors may be related to those of the electromagnetic electric and magnetic form factors of $^3\text{He}$ and $^3\text{H}$, or of the proton and the neutron. The value of the axial form factor $F_A$ at zero momentum transfer follows from the $\beta$-decay rate either of $^3\text{H}$ or of the neutron. Its value at nonvanishing momenta transfers requires knowledge of its $q^2$-dependency inferred from neutrino or pion electroproduction experiments. Finally, through PCAC, the value of the last first-class form factor, the induced pseudoscalar one, may be expressed as

$$F_{PCAC} = \frac{m_\mu (M_1 + M_2) F_A(q^2)}{m_\pi^2 - q^2} = \frac{m_\mu f_\pi g_{\pi N}(q^2)}{m_\pi^2 - q^2},$$

where $m_\pi$ is the $\pi^\pm$ mass, $f_\pi$ its decay constant, and $g_{\pi N}(q^2)$ its nuclear coupling to the two nuclear states of masses $M_1$ and $M_2$. In the limit of exact isospin symmetry, which implies also exact $G$-parity invariance, the values for the second-class form factors $F_S$ and $F_T$ may be shown to vanish identically. Hence, one expects deviations from zero for these form factors (normalized to $F_V$ or $F_A$) of only a few percent, as given by the ratio $(m_d - m_u)/\Lambda_{QCD}$ of $u$- and $d$-quark masses to the QCD scale for example, or the ratio $(m_n - m_p)/(m_n + m_p)$ in terms of the neutron and proton masses. Bag model or QCD sum rule evaluations of these form factors in the case of the (proton, neutron) doublet do indeed bear out such an expectation [38–41]. In the same manner, one could wonder about

To be precise, these expressions for $F_P$ assume implicitly that both $q^2$-dependencies for $F_A(q^2)$ and $g_{\pi N}(q^2)$ are identical [36]. Note also that through relations such as these, any experiment leading to a precise value for $F_P$ would also imply a precise value for the associated pion–nucleus coupling constant [37].
the effects of isospin symmetry breaking for the $F_V$ and $F_M$ form factors obtained through CVC. In this particular case, thanks to the Ademollo–Gatto theorem [42,43], isospin breaking corrections are only of quadratic order in the ratio $(m_d - m_u)/\Lambda_{\text{QCD}}$, hence negligible for our purposes. Let us also recall that CVC has been very well established through precision studies in $\beta$-decay [44], and that even though limits exist on second-class form factors from correlation experiments in $\beta$-decay, the stringency of these constraints are nowhere close to the theoretically expected values for $g_S$ or $g_T$ [35].

In the matrix elements for the scalar, pseudoscalar and tensor quark operators, we have chosen not to include recoil order induced contributions, for the reasons mentioned already in the Introduction. The values for the genuine scalar, pseudoscalar and tensor form factors $G_S$, $G_P$ and $G_T$, cannot be inferred from any experiment yet. One thus has to rely on specific model calculations for QCD dynamics, such as the bag model or QCD sum rules. However, no such results are available at present, and one may only reasonably guess that these form factors should take values on the order of unity, within a factor which at worst could be of order ten.

Hence, even though the parameterization of the nuclear matrix elements of the relevant quark operators in terms of form factors is only a phenomenological representation of our ignorance of the microscopic nonperturbative quark dynamics, this approach to nuclear muon capture allows nevertheless for explicit predictions independently of the details of nuclear models, relying only on the results of other experiments and the power of symmetry principles [6].

In terms of this parameterization of the nuclear state matrix elements, the effective muon capture amplitude at the nuclear level is given by

$$
\mathcal{M}_\lambda = \frac{g^2}{8M^2} V_{ud} \sum_{\eta_1, \eta_2 = +, -} \left[ \left( h_{\eta_1 \eta_2}^V \right)^* \bar{v}_\mu \gamma^\mu (1 + \eta_1 \gamma_5) \frac{1}{2M} \left( F_V + \eta_2 F_A \gamma_5 \right) \frac{q^\mu}{2M} \right] \psi_1
+ \frac{1}{2} \frac{1}{2M} \left( h_{\eta_1 \eta_2}^S \right)^* \bar{v}_\mu \sigma^{\mu\nu} (1 + \eta_1 \gamma_5) \frac{1}{2M} \left( G_S - \eta_2 G_P \gamma_5 \right) \psi_1 + \frac{\eta_2}{2M} \left( h_{\eta_1 \eta_2}^T \right)^* \bar{v}_\mu \sigma^{\mu\nu} (1 + \eta_1 \gamma_5) \frac{1}{2M} \left( G_T - \eta_2 G_P \gamma_5 \right) \psi_1 + \frac{1}{2} \frac{1}{2M} \left( h_{\eta_1 \eta_2}^T \right)^* \bar{v}_\mu \sigma^{\mu\nu} (1 + \eta_1 \gamma_5) \frac{1}{2M} \left( G_T - \eta_2 G_P \gamma_5 \right) \psi_1 \right]. (10)
$$

2.3. The method of calculation

The remainder of the calculation requires now the evaluation of $|\mathcal{M}_\lambda|^2$. If one were to proceed by “brute force”, using the usual trace techniques for such calculations, one would quickly run into unmanageable expressions, because of the many contributions stemming from all the interactions represented in the amplitude $\mathcal{M}_\lambda$. Actually, it is possible in the
In the present case to take advantage of the fact that the initial system is at rest, and that to a very good approximation the initial muon and nucleus may be considered to be also at rest. For that purpose, it turns out that the Dirac representation of the Dirac–Clifford algebra of $\gamma^\mu$ matrices is the best suited for the problem. Plane wave solutions to the massive Dirac equation are then of the following form, for “positive” and “negative” energy solutions, respectively (our Minkowski metric signature convention is $(-++)$):

$$u(\vec{k}, \tilde{\sigma}) = \frac{1}{\sqrt{k^0 + m}} (k^\mu \gamma_\mu + m) \begin{pmatrix} \chi_+(\tilde{\sigma}) \\ 0 \end{pmatrix},$$

$$v(\vec{k}, \tilde{\sigma}) = \frac{1}{\sqrt{k^0 + m}} (-k^\mu \gamma_\mu + m) \begin{pmatrix} 0 \\ \chi_-(\tilde{\sigma}) \end{pmatrix},$$

where $\chi_{\pm}(\tilde{\sigma})$ are bi-spinors such that

$$\tilde{\sigma} \cdot \tilde{\sigma} \chi_{\pm}(\tilde{\sigma}) = \pm \chi_{\pm}(\tilde{\sigma}), \quad \chi_{\pm}^\dagger(\tilde{\sigma}) \chi_{\pm}(\tilde{\sigma}) = 1,$$

with $\sigma^i (i = 1, 2, 3)$ the usual Pauli matrices, while $\tilde{\sigma}$ is a unit vector in three dimensions which in fact corresponds to the spin polarization vector of the particle (see Appendix B).

Further properties of these bi-spinors are

$$\chi_{+}(-\tilde{\sigma}) = i\chi_{-}(\tilde{\sigma}), \quad \chi_{-}(-\tilde{\sigma}) = i\chi_{+}(\tilde{\sigma}),$$

$$\chi_0(\tilde{\sigma}) \chi_0^\dagger(\tilde{\sigma}) = \frac{1}{2} [1 + \eta \tilde{\sigma} \cdot \tilde{\sigma}], \quad \eta = \pm.$$

In fact, it is only the latter relation which is of essential use in the calculation of $|\mathcal{M}_\lambda|^2$.

Even though this is not important for that calculation, it is also possible to give the explicit expressions for these bi-spinors. Associated to the spherical angular parameterization of the unit vector $\tilde{\sigma}$,

$$\tilde{\sigma} = \begin{pmatrix} \sin \theta \cos \phi \\ \sin \theta \sin \phi \\ \cos \theta \end{pmatrix},$$

one has

$$\chi_{\pm}(\tilde{\sigma}) = \begin{pmatrix} e^{-i\varphi/2} \cos \theta/2 \\ e^{i\varphi/2} \sin \theta/2 \end{pmatrix}, \quad \chi_{-}(\tilde{\sigma}) = \begin{pmatrix} -e^{-i\varphi/2} \sin \theta/2 \\ e^{i\varphi/2} \cos \theta/2 \end{pmatrix}. \quad (16)$$

Finally in the case of solutions to the massless Dirac equation, “positive” and “negative” energy plane wave solutions of helicity $\lambda = \pm 1$ and momentum $\vec{k}$ are given by

$$u(\vec{k}, \lambda) = \sqrt{k^0} \begin{pmatrix} \chi_\lambda(\hat{k}) \\ \lambda \chi_\lambda(\hat{k}) \end{pmatrix}, \quad v(\vec{k}, \lambda) = \sqrt{k^0} \begin{pmatrix} \chi_\lambda(\hat{k}) \\ \lambda \chi_\lambda(\hat{k}) \end{pmatrix},$$

where of course $\hat{k} = \vec{k} / |\vec{k}|$. 

9 This assumption amounts to ignoring those small relativistic corrections which are related to the velocities of the bound muon and nucleus, which in the present case is indeed totally justified. A fully satisfactory relativistic treatment of such a bound state problem in QED is still not available, and cannot be used here to estimate small corrections which in any case will be at most of order $(a Z)^2$, namely the squared velocity of the bound muon.
In order to apply these considerations to the capture amplitude (10), let us denote by $\chi_\mu(\tilde{\sigma}_\mu)$, $\chi_1(\tilde{\sigma}_1)$, $\chi_2(\tilde{\sigma}_2)$ and $\chi_\lambda(-\tilde{p})$ the bi-spinors associated to the muon, the initial and final nuclei, and the neutrino, respectively (for the first three states, they thus correspond to bi-spinors of $\chi_\mu(\tilde{\sigma})$ type). After substitution in (10), the capture amplitude associated to a massless neutrino of helicity $\lambda$ then reduces to the expression:

$$\mathcal{M}_\lambda = \left(\frac{g^2}{8M^2}\right)V_{ud} \sqrt{4m M_1} \sqrt{\frac{v}{2\sqrt{s}}} \times 2\left\{ \chi_\lambda^\dagger \chi_2 \left[ H^{(S)}_\lambda + (H^{(P)}_\lambda - H^{(A)}_\lambda) \tilde{p} \cdot \sigma \right] \chi_1 + \lambda \chi_\lambda^\dagger \sigma^i \chi_2 \left[ H^{(V)}_\lambda \tilde{\nu} \tilde{p} \times \sigma^j - H^{(A)}_\lambda \sigma^i \right] \chi_1 \right\},$$

(18)

with coefficients $H^{(S,P,V,A)}_\lambda$ defined by the following relations:

$$H^{(S)}_\lambda = \left( \sum_{\eta_2 = \pm} h^{V}_{\lambda,\eta_2} \right)^* G^{(1)}_V + \left( \sum_{\eta_2 = \pm} h^{S}_{\lambda,\eta_2} \right)^* G^{(1)}_S - 2(h^{T}_{-\lambda\lambda})^* G^{(1)}_T,$$

(19)

$$H^{(P)}_\lambda = \left( \sum_{\eta_2 = \pm} h^{V}_{\lambda,\eta_2} \right)^* G^{(1)}_A + \left( \sum_{\eta_2 = \pm} h^{S}_{\lambda,\eta_2} \right)^* G^{(1)}_P - 2\lambda(h^{T}_{-\lambda\lambda})^* G^{(2)}_T,$$

(20)

$$H^{(V)}_\lambda = \left( \sum_{\eta_2 = \pm} h^{V}_{\lambda,\eta_2} \right)^* G^{(2)}_V - 2(h^{T}_{-\lambda\lambda})^* G^{(1)}_T,$$

(21)

$$H^{(A)}_\lambda = \left( \sum_{\eta_2 = \pm} h^{V}_{\lambda,\eta_2} \right)^* G^{(2)}_A - 2\lambda(h^{T}_{-\lambda\lambda})^* G^{(2)}_T,$$

(22)

in which the following combinations of form factors are introduced:

$$G^{(1)}_V = (\sqrt{s} - M_2) \left[ F_V - \frac{\sqrt{s} - M_1}{2M} F_M \right] + (\sqrt{s} + M_2) \left[ F_V + \frac{\sqrt{s} - M_1}{2M} F_S \right],$$

(23)

$$G^{(1)}_A = (\sqrt{s} - M_2) \left[ F_A - \frac{\sqrt{s} - M_1}{M_\mu} F_P \right] + (\sqrt{s} + M_2) \left[ F_A + \frac{\sqrt{s} - M_1}{2M} F_T \right],$$

(24)

$$G^{(2)}_V = (\sqrt{s} - M_2) \left[ F_V + \frac{M_1 + M_2}{2M} F_M \right],$$

(25)

$$G^{(2)}_A = (\sqrt{s} + M_2) \left[ F_A - \frac{M_1 - M_2}{2M} F_T \right],$$

(26)

$$G^{(1)}_S = (\sqrt{s} + M_2) G_S,$$

(27)

$$G^{(1)}_P = (\sqrt{s} - M_2) G_P,$$

(28)

$$G^{(1)}_T = (\sqrt{s} - M_2) G_T,$$

(29)

$$G^{(2)}_T = (\sqrt{s} + M_2) G_T.$$  

(30)

The calculation of $|\mathcal{M}_\lambda|^2$ then proceeds using the trace properties in (14). The advantage of the above approach is that the combinations of form factors and coupling coefficients which are relevant appear from the start, ever before proceeding to the calculation of traces. Were one to first calculate the traces as is usual, the task would quickly become impossible.
in the general case considered here (presumably, this is the reason why only numerical expressions were given in Ref. [23], even though the situation considered there was far less general). Still, using the present approach, the length of the calculation is of some importance. Note also that no nonrelativistic expansion in the amplitude is effected at any stage of the calculation, in contradistinction to all other analyses which are based on Ref. [6] in which such an expansion in $1/M$ is indeed applied.

Finally, the capture distribution is thus given by

$$\frac{d\Gamma}{d\Omega_p} = \frac{|\psi(0)|^2}{32\pi^2} \left( \frac{s^2}{8M^2} \right)^2 \frac{v^2}{s} R_\lambda \quad \text{with} \quad (31)$$

$$R_\lambda = (1 - \lambda \hat{p} \cdot \hat{s}_\mu) \{ C_\lambda^{(\mu)} (1 + (\hat{p} \cdot \hat{s}_1)(\hat{p} \cdot \hat{s}_2)) + D_\lambda^{(\mu)} ((\hat{s}_1 \cdot \hat{s}_2) - (\hat{p} \cdot \hat{s}_1)(\hat{p} \cdot \hat{s}_2) + E_\lambda^{(\mu)} \hat{p} \cdot (\hat{s}_1 \times \hat{s}_2) \}$$

$$+ (1 - \lambda \hat{p} \cdot \hat{s}_1) \{ C_\lambda^{(1)} (1 + (\hat{p} \cdot \hat{s}_\mu)(\hat{p} \cdot \hat{s}_2)) + D_\lambda^{(1)} ((\hat{s}_\mu \cdot \hat{s}_2) - (\hat{p} \cdot \hat{s}_\mu)(\hat{p} \cdot \hat{s}_2) + E_\lambda^{(1)} \hat{p} \cdot (\hat{s}_\mu \times \hat{s}_2) \}$$

$$+ (1 + \lambda \hat{p} \cdot \hat{s}_2) \{ C_\lambda^{(2)} (1 - (\hat{p} \cdot \hat{s}_\mu)(\hat{p} \cdot \hat{s}_1)) + D_\lambda^{(2)} ((\hat{s}_\mu \cdot \hat{s}_1) - (\hat{p} \cdot \hat{s}_\mu)(\hat{p} \cdot \hat{s}_1)) + E_\lambda^{(2)} \hat{p} \cdot (\hat{s}_\mu \times \hat{s}_1) \}, \quad (32)$$

in which the following final definitions apply:

$$C_\lambda^{(\mu)} = |H_\lambda^{(S)}|^2 + |H_\lambda^{(P)}|^2 - 2|H_\lambda^{(V)} + \lambda H_\lambda^{(A)}|^2, \quad (33)$$

$$C_\lambda^{(1)} = -2\lambda \Re \left( H_\lambda^{(S)} \left( H_\lambda^{(P)} \right) ^* \right) + 2\left| H_\lambda^{(V)} + \lambda H_\lambda^{(A)} \right|^2, \quad (34)$$

$$C_\lambda^{(2)} = 2\lambda \Re \left( H_\lambda^{(S)} \left( H_\lambda^{(P)} \right) ^* \right) + 2\left| H_\lambda^{(V)} + \lambda H_\lambda^{(A)} \right|^2, \quad (35)$$

$$D_\lambda^{(\mu)} = \left| H_\lambda^{(S)} \right|^2 - \left| H_\lambda^{(P)} \right|^2, \quad (36)$$

$$D_\lambda^{(1)} = -2 \Re \left( \left( H_\lambda^{(S)} - \lambda H_\lambda^{(P)} \right) \left( H_\lambda^{(V)} + \lambda H_\lambda^{(A)} \right) ^* \right), \quad (37)$$

$$D_\lambda^{(2)} = -2 \Re \left( \left( H_\lambda^{(S)} + \lambda H_\lambda^{(P)} \right) \left( H_\lambda^{(V)} + \lambda H_\lambda^{(A)} \right) ^* \right), \quad (38)$$

$$E_\lambda^{(\mu)} = 2 \Im \left( H_\lambda^{(S)} \left( H_\lambda^{(P)} \right) ^* \right), \quad (39)$$

$$E_\lambda^{(1)} = -2 \lambda \Im \left( \left( H_\lambda^{(S)} - \lambda H_\lambda^{(P)} \right) \left( H_\lambda^{(V)} + \lambda H_\lambda^{(A)} \right) ^* \right), \quad (40)$$

$$E_\lambda^{(2)} = -2 \lambda \Im \left( \left( H_\lambda^{(S)} + \lambda H_\lambda^{(P)} \right) \left( H_\lambda^{(V)} + \lambda H_\lambda^{(A)} \right) ^* \right). \quad (41)$$

Note that these results show that $T$-odd effects can only appear through the triple correlation coefficients of $E_\lambda^{(\mu,1,2)}$ type, related to contributions which are pure imaginary under complex conjugation and involving necessarily always at least two polarization vectors.

2.4. Capture distributions and final state polarization

The calculation thus leads to the capture distribution as given in (32). For a massless neutrino of given helicity $\lambda$, (i.e. a Weyl neutrino, or a neutrino whose helicity is measured!), (32) gives the final expression. For a massless Dirac neutrino produced with either helicity (which is then not measured), one needs still to sum the result over $\lambda = \pm 1$. Finally, and
independently of the neutrino sector, in order to obtain the unpolarised nuclear final state
distribution, one simply needs to sum the two results obtained from (32) for \( \hat{\sigma}_2 = \hat{\sigma}_0 \) and
\( \hat{\sigma}_2 = -\hat{\sigma}_0 \). Since the result (32) is at most linear in \( \hat{\sigma}_2 \), this amounts to setting \( \hat{\sigma}_2 = 0 \) and to
multiply the result (32) by a factor two.

In general, (32) thus has the parameterization
\[
R_\lambda = N_\lambda + \hat{\sigma}_2 \cdot \bar{D}_\lambda, \tag{42}
\]
where the expressions for the coefficients \( N_\lambda \) and \( \bar{D}_\lambda \) may be read off directly from (32).
The capture distribution of the final state nucleus is thus given by
\[
\frac{dF_\lambda}{d\Omega_\rho} = \frac{|\psi(0)|^2}{32\pi^2} \left( \frac{g^2}{8M^2} \right)^2 V_{nd}^2 \frac{v^2}{s} 2N_\lambda, \tag{43}
\]
while the final state spin 1/2 nucleus has the following average polarization vector,
\[
\langle \hat{\sigma}_2 \rangle_\lambda = \frac{1}{N_\lambda} \bar{D}_\lambda. \tag{44}
\]

For example, taking an initially unpolarized state with \( \hat{\sigma}_\mu = \hat{\sigma}_1 = 0 \), the average
polarization of the final nucleus is given by
\[
\langle \hat{\sigma}_2 \rangle_\lambda^{\text{unpolarized}} = \lambda \frac{C_\lambda^{(2)}}{C_\lambda^{(1)} + C_\lambda^{(1)} + C_\lambda^{(2)}} \hat{p}. \tag{45}
\]

Hence, a transversally polarized final nucleus requires an initially polarized muonic atom.

However, all the results discussed so far refer to polarization states defined in terms of
the individual polarization vectors \( \hat{\sigma}_\mu, \hat{\sigma}_1, \) and \( \hat{\sigma}_2 \), rather than the hyperfine states \( (S, m) \)
of the initial muonic atom, with \( (S = 1, m = \pm 1, 0) \) and \( (S = 0, m = 0) \). As explained
in Appendix B, it is of course possible to use the above results to determine the final
state distributions and polarizations associated to each of these hyperfine states. Since it
is difficult to imagine how the final state polarization of the neutron or \(^3\text{H}\) could ever be
measured to any degree of precision, here only the relevant expressions for the hyperfine
capture distributions are presented.

Given the hyperfine states \( S = 0, 1 \) and their projections \( m = 0 \) and \( m = 0, \pm 1 \) along
some arbitrary quantization axis, the associated hyperfine capture distributions are given by
\[
\frac{dF_\lambda^{(S,m)}}{d\Omega_\rho} = \frac{|\psi(0)|^2}{16\pi^2} \left( \frac{g^2}{8M^2} \right)^2 V_{nd}^2 \frac{v^2}{s} R_\lambda^{(S,m)}, \tag{46}
\]
where the usual Legendre polynomials are
\[
P_1(\cos \theta) = \cos \theta, \quad P_2(\cos \theta) = \frac{3}{2} \cos^2 \theta - \frac{1}{2}. \tag{50}
\]
while \( \theta \) is the angle between the spin quantization axis and the normalized momentum vector \( \vec{p} \) of the outgoing spin 1/2 nucleus.

In particular, the statistical capture distribution is given as in (46) with the coefficient \( R^\text{stat}_\lambda \), then obtained from

\[
R^\text{stat}_\lambda = \frac{1}{4} \left[ R^{(1,+1)}_\lambda + R^{(1,-1)}_\lambda + R^{(1,0)}_\lambda + R^{(0,0)}_\lambda \right] \\
= C^{(\mu)}_\lambda + C^{(1)}_\lambda + C^{(2)}_\lambda = |H^{(S)}_\lambda|^2 + |H^{(P)}_\lambda|^2 + 2 |H^{(V)}_\lambda + \lambda H^{(A)}_\lambda|^2,
\]

a result which is indeed \( \theta \)-independent as it should.

Integrating these distributions leads to the associated capture rates, \( \lambda^S_\lambda, \lambda^T_\lambda \) and \( \lambda^\text{stat}_\lambda \) for the singlet \((S)\), triplet \((T)\) and statistical rates, respectively, of the form

\[
\lambda^S,T,\text{stat}_\lambda = \frac{|\psi(0)|^2}{4\pi} \left( \frac{s^2}{8M^2} \right)^2 V^2_\nu \frac{v^2}{s} R^S,T,\text{stat}_\lambda,
\]

with \( R^\text{stat}_\lambda \) for the statistical capture rate given in (51) already, while for the singlet and triplet capture rates, one has, respectively:

\[
R^S_\lambda = C^{(\mu)}_\lambda + C^{(1)}_\lambda + 2 C^{(2)}_\lambda - 2 D^{(2)}_\lambda = R^{(0,0)}_\lambda, \\
R^T_\lambda = C^{(\mu)}_\lambda + C^{(1)}_\lambda + \frac{3}{2} [C^{(2)}_\lambda + D^{(2)}_\lambda].
\]

It is also possible to represent the hyperfine capture distribution (46) in the following form [4,5]:

\[
\frac{d\Gamma_\lambda}{d\Omega_\vec{p}} = \frac{1}{4\pi} \lambda^\text{stat}_\lambda \left[ 1 + A^\Delta_\lambda P_\Delta + A^i_\lambda P_i \cos \theta + A^t_\lambda P_t \left( \frac{1}{2} \cos^2 \theta - \frac{1}{2} \right) \right],
\]

where \( A^\Delta_\lambda, A^i_\lambda \) and \( A^t_\lambda \) are specific coefficients, the latter two known as the vector and tensor analyzing powers of the final state nucleus, respectively, while \( A^\Delta_\lambda \) is a measure of the hyperfine effect on the statistical capture rate since one has

\[
A^\Delta_\lambda = \frac{1}{4} \frac{\lambda^T_\lambda - \lambda^S_\lambda}{\lambda^\text{stat}_\lambda} = \frac{\lambda^T_\lambda - 3 \lambda^S_\lambda}{4 \lambda^\text{stat}_\lambda}.
\]

Finally in (55), the coefficients \( P_{\Delta,v,t} \) are the following combinations of the hyperfine populations \( N_{S,m} \):

\[
P_\Delta = N_{1,1} + N_{1,0} + N_{1,-1} - 3N_{0,0} = 1 - 4N_{0,0},
\]

\[
P_v = N_{1,1} - N_{1,-1}, \quad P_t = N_{1,1} + N_{1,-1} - 2N_{1,0},
\]

such that \( N_{1,1} + N_{1,-1} + N_{1,0} + N_{0,0} = 1 \).

In terms of the quantities introduced above, one then finds:

\[
A^\Delta_\lambda = - \frac{1}{3} \frac{C^{(2)}_\lambda - 2 D^{(2)}_\lambda}{R^\text{stat}_\lambda}, \quad A^i_\lambda = \frac{\lambda (C^{(\mu)}_\lambda + C^{(1)}_\lambda)}{R^\text{stat}_\lambda}, \quad A^t_\lambda = - \frac{2}{3} \frac{C^{(2)}_\lambda + D^{(2)}_\lambda}{R^\text{stat}_\lambda}.
\]

Note that when observables are considered in which the summation over the two neutrino helicity states \( \lambda = \pm 1 \) has been effected, this summation has to be applied separately in the numerator and the denominator of each of the expressions given in this subsection.
3. The $^3$He case

3.1. Physical inputs

The basic kinematical input used in the case of muon capture on $^3$He is as follows:

\[
\begin{align*}
M_1 &= 2808.392\text{ MeV}, & M_2 &= 2808.928\text{ MeV}, & \sqrt{s} &= 2914.039\text{ MeV}, \\
\nu &= 103.22\text{ MeV}, & \omega &= 2810.82\text{ MeV}, & \omega - M_2 &= 1.90\text{ MeV}.
\end{align*}
\]  

(60)

As explained in Appendix A, the overlap reduction factor $C$ in this case takes the value $C_D = 0.979$ [4,5], while the latest value quoted in Ref. [35] for the CKM $ud$ mixing angle is used, namely:

\[
V_{ud} = 0.9750 \pm 0.0008.
\]  

(61)

This value results from a global fit which includes the constraints of unitarity of the CKM mixing matrix, rather than the value which would follow solely from the $0^+ - 0^+$ superallowed $\beta$-decay $f_i$-values.

Values for the nuclear form factors at the relevant momentum transfer $q_1^2 = -0.954m_\mu^2$ must also be specified. For this purpose, we refer to the discussion in Refs. [4,5] and use the values advocated by these authors. For the electroweak vector and induced magnetic form factors, one has:

\[
\begin{align*}
F_V(q_1^2) &= 0.834 \pm 0.011, & F_M(q_1^2) &= -13.969 \pm 0.052.
\end{align*}
\]  

(62)

The situation for $F_A(q_1^2)$ is somewhat more delicate, since only its value at $q^2 = 0$ is known from $\beta$-decay of $^3$H, while its $q^2$-dependency may only be inferred through a nuclear model calculation of the associated mean square charge radius [4,5]. This leads to the following value:

\[
F_A(q_1^2) = -1.052 \pm [0.005 - 0.01].
\]  

(63)

where the interval for the uncertainty is an attempt to reflect the lack of precise knowledge of this form factor. The lower uncertainty of 0.005 stems from the uncertainties on the $^3$H $\beta$-decay rate, while the corrections due to mesonic exchange currents may only be estimated in the nuclear model calculation, leading to a total uncertainty of 0.007 on $F_A(q_1^2)$ [4,5]. Thus the bracket $[0.005 - 0.01]$ represents a conservative evaluation of the uncertainty on that form factor, which will be carried along throughout our analysis later on. As it turns out, the statistical capture rate is rather sensitive to that quantity, so that any improvement on its uncertainty leads to an improvement on the values of other quantities that one infers from experiment. However, it is difficult to see how such an improvement on $F_A(q_1^2)$ could be achieved in practice, since it would require a measurement of the $q^2$-dependency of the $^3$He-$^3$H axial form factor $F_A(q_1^2)$.

\footnote{The uncertainty on this value is irrelevant for the $^3$He case, but must be included in the prediction for muon capture on the proton at the precision level reached in Section 4.2. Note also that this value for $V_{ud}$ differs somewhat from that used previously [17–19], which implies that the numbers quoted here differ somewhat from those quoted earlier, but in no physically significant way whatsoever.}
When it has to be specified, the value used for the induced pseudoscalar form factor \( F_p \) is that inferred from the PCAC relation (9), which corresponds to

\[
F_p^{\text{PCAC}}(q_1^2) = -20.73 \pm [0.10-0.20],
\]

where the uncertainty range indicated in brackets thus corresponds to the associated range in the value used for \( F_A(q_1^2) \).

Finally, the remaining two second-class form factors for the vector and axial quark operators, \( F_V(q_1^2) \) and \( F_T(q_1^2) \), are taken to be vanishing, even though for the reasons explained in Section 2.2, one should expect that their absolute values, normalized to those for \( F_V(q_1^2) \) and \( F_A(q_1^2) \), respectively, would be on the order of 0.02. As will become clear later on, the statistical capture rate is in fact rather insensitive to these two form factors, so that this reasonable approximation is certainly sufficient.

These are all the nuclear form factors required when considering muon capture within the SM. Beyond that model however, the genuine nuclear scalar, pseudoscalar and tensor form factors, \( G_S(q_1^2), G_P(q_1^2) \) and \( G_T(q_1^2) \), are also required. As mentioned previously, values for these quantities are not known, but are expected to be on the order of unity within a factor of at most ten. Nevertheless, any constraints put on the effective coupling coefficients \( h_{S,V,T} \) beyond the SM will thus involve at present these nuclear form factors as well.

### 3.2. Within the standard model

Capture within the SM corresponds to the amplitude \( M_\lambda \) in (10) with only the \( h_{V,\lambda} = 1 \) effective coupling coefficient turned on. Given the above numerical values, as well as the general expressions of Section 2.4, it is straightforward to determine theoretical values for all the quantities which enter the general final state triton distribution associated to hyperfine states as parameterized in (55). One finds:

\[
\begin{align*}
\lambda^S & = 1929 \pm [28-46] \text{ s}^{-1}, & A_\Delta & = -0.0967 \pm [0.0061-0.0069], \\
\lambda^T & = 1351 \pm [16-19] \text{ s}^{-1}, & A_v & = +0.524 \pm [0.0057-0.0061], \\
\lambda^{\text{stat}} & = 1496 \pm [12-21] \text{ s}^{-1}, & A_t & = -0.3790 \pm [0.00074-0.00123],
\end{align*}
\]

where each time the indicated uncertainty includes all uncertainties of all the theoretical input, while the range indicated by the bracket corresponds to the range implied by the uncertainty on \( F_A(q_1^2) \). Obviously, some of these results coincide with the specific quantities also computed in Refs. [4,5] in this particular case.

The disparity in the uncertainty ranges for these quantities stems from their sensitivity to all theoretical inputs. This sensitivity of an observable \( O \) to a parameter \( F_X \) may be specified in terms of the variations, evaluated in the SM,

\[
\sigma(O; F_X) = \frac{F_X}{O} \frac{\partial O}{\partial F_X|_{\text{SM}}} \text{ if } F_X \neq 0; \quad \sigma(O; F_X) = \frac{1}{O} \frac{\partial O}{\partial F_X|_{\text{SM}}} \text{ if } F_X = 0.
\]

\[\text{Note that the helicity index } \lambda \text{ is suppressed, since whether one sums or not over the helicities of the final neutrino which is only of left-handed chirality is irrelevant in the massless limit in the present case.}\]
Table 1
Sensitivities of the different observables $O = \lambda^{S,T,\text{stat}}$, $A_{\Delta}$, $A_v$, $A_t$ for muon capture on $^3$He, with respect to the nuclear form factors associated to the vector and axial quark operators. See (66) for the definition of $\sigma(O; F_X)$.

<table>
<thead>
<tr>
<th>$F_V$</th>
<th>$F_M$</th>
<th>$F_A$</th>
<th>$F_P$</th>
<th>$F_S$</th>
<th>$F_T$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-0.726$</td>
<td>$+0.420$</td>
<td>$+2.621$</td>
<td>$-0.314$</td>
<td>$-0.0155$</td>
<td>$-0.0157$</td>
</tr>
<tr>
<td>$+0.790$</td>
<td>$+0.233$</td>
<td>$+0.997$</td>
<td>$-0.0210$</td>
<td>$+0.0178$</td>
<td>$-0.00112$</td>
</tr>
<tr>
<td>$+0.301$</td>
<td>$+0.293$</td>
<td>$+1.521$</td>
<td>$-0.116$</td>
<td>$+0.0071$</td>
<td>$-0.0058$</td>
</tr>
<tr>
<td>$-4.572$</td>
<td>$+0.561$</td>
<td>$+4.894$</td>
<td>$-0.884$</td>
<td>$-0.101$</td>
<td>$-0.044$</td>
</tr>
<tr>
<td>$+0.798$</td>
<td>$-0.287$</td>
<td>$-0.134$</td>
<td>$-0.377$</td>
<td>$+0.0173$</td>
<td>$-0.0185$</td>
</tr>
<tr>
<td>$+0.0616$</td>
<td>$+0.254$</td>
<td>$-1.063$</td>
<td>$+0.747$</td>
<td>$+0.0017$</td>
<td>$+0.0368$</td>
</tr>
</tbody>
</table>

The values of these quantities for all the above hyperfine observables are shown in Table 1. Note that except for $A_v$, all these observables are rather sensitive to the axial form factor $F_A(q^2)$, and thus to its bracket of uncertainty. Sensitivities to $F_P(q^2)$ include the results of Refs. [4,5], showing in fact that except for the triplet capture rate, the statistical rate which has been measured [14] is the least sensitive observable to that form factor, with the obvious drawback that all other observables are still far more difficult to measure.

Given the experimental result for the statistical capture rate [14],

$$\lambda^{\text{stat}}_{\text{exp}} = 1496 \pm 4 \text{ s}^{-1},$$  

(67)

it is possible to infer a value for any given parameter once the values for all other quantities are specified. Thus given the inputs discussed in Section 3.1 except for the value for $F_P(q^2)$, the result (67) implies [14,16–19]:

$$F_P(q^2) = -20.69 \pm [1.57 – 2.74] [\text{exp: 0.48}],$$  

(68)

where the first uncertainty bracket includes both the experimental error as well as all the uncertainties on the input form factors, the range corresponding again to the range in the $F_A(q^2)$ uncertainty, while the last number in brackets represents the uncertainty following only from the experimental error in (67), namely without any of the errors on the other inputs. This last number thus indicates the range of improvement that could be achieved by reducing the uncertainties on the input form factors $F_V(q^2)$, $F_M(q^2)$ and $F_A(q^2)$, and especially on the latter one. Compared to the PCAC expected value for this induced pseudoscalar form factor, the above result is thus in confirmation of the PCAC prediction with a precision ranging from 8% to 13%. In order to translate this conclusion in terms of the nucleon form factor $g_P$, it is necessary to include in the nuclear model calculation all meson exchange corrections and their associated uncertainties, thereby leading to a 19% precise test of PCAC with a value in agreement with the prediction [15]. Note also that given the PCAC relation (9), the above value for $F_P(q^2)$ inferred from experiment may in turn be used to determine the $\pi^\pm - ^3\text{He} - ^3\text{H}$ nuclear coupling constant to much improved precision [37].

A combined fit to two independent observables measured to great precision, such as $\lambda^{\text{stat}}$ and the vector analyzing power $A_v$, would allow a model independent determination...
of both $F_A(q_1^2)$ and $F_P(q_1^2)$. However, it may be shown that in order to obtain a result for $F_A(q_1^2)$ with an uncertainty less than the present value of 0.01, would require a measurement of $A_v$ to better than 1%, no small feat indeed! In particular, when used on its own, a 1% precise measurement of $A_v$, centered onto its theoretical prediction, would imply the following uncertainty range for $F_P(q_1^2)$ when using for the other form factors the values quoted in Section 3.1:

$$F_P(q_1^2) = 0.026 \pm [1.17-2.02][\exp: 0.38], \quad \text{or}$$

$$F_T(q_1^2) = -0.031 \pm [1.42-2.45][\exp: 0.46],$$

when either form factor is turned on while the other is still set equal to zero. Due to the small sensitivity of the statistical capture rate to these two parameters, these constraints are thus extremely poor, but nevertheless they improve somewhat the situation existing in terms of the nucleon form factors $g_S$ and $g_T$ \[25\] if one is willing to extrapolate without correction from the hydrogen case. This conclusion would not be much improved were a 1% precise measurement of $A_v$ to become available, since the corresponding uncertainties ranges are then

$$F_S(q_1^2) = [0.80-0.80][\exp: 0.55].$$

(69)

Turning now to the second-class form factors $F_S(q_1^2)$ and $F_T(q_1^2)$, and using the PCAC prediction (64) for $F_P(q_1^2)$, the experimental result (67) implies either

$$F_S(q_1^2) = 0.026 \pm [1.17-2.02][\exp: 0.38], \quad \text{or}$$

$$F_T(q_1^2) = -0.031 \pm [1.42-2.45][\exp: 0.46].$$

(70)

(71)

Similarly considerations could of course be developed on the basis of other observables still, but we shall refrain from doing so since they are beyond the reach of experiment at present.

Finally, let us point out that if $F_P(q_1^2)$ is allowed to vary within its uncertainty bracket in (68), the corresponding values for $F_S(q_1^2)$ and $F_T(q_1^2)$ then also vary accordingly, but still within their respective uncertainty brackets. Hence any dependency of the results for $F_S,T$ on the value assumed for $F_P$ is consistent with their own present uncertainties.

### 3.3. Beyond the standard model

Let us now turn to the reach for physics beyond the SM offered by the experimental result (67). Defining sensitivities $\sigma(O; h_X)$ of observables to the effective coupling coefficients $h_{S,V,T}$ in the same manner as in (66) with respect to the vanishing second-class form factors $F_S,T$, the corresponding results are given in Table 2, using as input for nuclear form factors the values discussed in Section 3.1 as well as unit values for the scalar, pseudoscalar and tensor form factors $G_{S,P,T}$. In fact, these observables are not sensitive (in linear order) to the coefficients $h_\pm, h^\pm_T$ and $h^{V,\pm,\pm}_T$ because of the fact that only a left-handed neutrino couples to muon capture in the limit of the SM. Note that some of these observables are quite sensitive to the tensor coupling $h_{-,+}^T$, for which stringent constraints will thus be inferred from (67).
Indeed, the experimental statistical capture rate (67) implies the following results, turning on only one coupling at a time:

\[ h_{+}^{S} = 0.00049 \pm [0.0224-0.0385] \text{(exp: 0.00723)}, \]
\[ h_{+}^{T} = 0.00047 \pm [0.0218-0.0374] \text{(exp: 0.00702)}, \]
\[ h_{+}^{P} = 0.00048 \pm [0.0221-0.0379] \text{(exp: 0.00712)}, \]
\[ h_{+}^{\Delta} = -0.0322 \pm [1.49-2.55] \text{(exp: 0.48)}, \]
\[ h_{-}^{S} = 0.00031 \pm [0.00143-0.00245] \text{(exp: 0.00046)}, \]
\[ h_{-}^{T} = 1.0009 \pm [0.00415-0.00712] \text{(exp: 0.00134)}, \]
\[ h_{-}^{P} = -0.000222 \pm [0.0102-0.0176] \text{(exp: 0.0033)}. \]

These values assume implicitly that the form factors \(G_{S, P, T}\) have been set equal to unity. Note also that the coefficients \(h_{+}^{S}\) and \(h_{+}^{T}\) do define actual scalar and pseudoscalar interactions at the quark-lepton level, for a neutrino of left-handed chirality.

Again, it may be checked that allowing \(F_{P}\) to vary within its uncertainty bracket (68), each of the above coefficients then also varies essentially within its own uncertainty bracket. As could be expected from Table 2, the results for the scalar couplings \(h_{+}^{S}\), \(h_{+}^{T}\), and \(h_{+}^{P}\) are already quite stringent, and in fact improve present limits on such couplings both in the muonic as well as in electronic sectors. However, the most satisfactory result is undoubtedly obtained for the tensor coupling \(h_{+}^{T}\), which is brought down into the per mille level. Of course, definite conclusions as to the actual limits for such physics beyond the SM implied by the experimental result (67) would require the evaluation of the form factors \(G_{S}(q_{1}^{2})\), \(G_{P}(q_{1}^{2})\) and \(G_{T}(q_{1}^{2})\). The above limits on \(h_{+}^{S, V, T}\) are translated in what may be physically more meaningful terms in Section 5.
Again for comparison, it is interesting to establish what a 1% precise measurement of \( A_{\nu} \), centered onto its theoretical prediction, would imply for the same set of effective coupling coefficients. Correspondingly, one finds for the associated uncertainty brackets:

\[
\begin{align*}
    h_{CV T}^\Sigma &\approx 0.164 - 0.0170 [\text{exp: 0.0111}], \\
    h_{CV T}^{\Sigma+} &\approx 0.157 - 0.0164 [\text{exp: 0.0107}], \\
    h_{CV T}^\Sigma &\approx 0.161 - 0.0167 [\text{exp: 0.0109}], \\
    h_{CV T}^P &\approx 0.81 - 0.84 [\text{exp: 0.55}], \\
    \frac{1}{2}h_{CV T}^{\Sigma+} &\approx 0.0201 - 0.0210 [\text{exp: 0.0136}], \\
    h_{CV T}^V &\approx 0.144 - 0.0150 [\text{exp: 0.0098}].
\end{align*}
\] (74)

Hence, such results would improve somewhat on the situation for some coefficients in (73) following from the experimental statistical capture rate (67).

4. The hydrogen case

4.1. Physical inputs

In the case of muon capture on the proton, the kinematic variables are such that we have

\[
\sqrt{s} = 1043.93 \text{ MeV}, \quad \nu = 99.146 \text{ MeV}, \\
\omega = 944.78 \text{ MeV}, \quad \omega - M_2 = 5.22 \text{ MeV}.
\] (75)

We also use for the CKM \( u d \) matrix element the value for \( V_{ud} \) quoted in (61), including its uncertainty. Indeed, the latter contributes 0.16% to the final uncertainty on the theoretical prediction, and should thus be included given the aim of a 0.5% precise measurement of the singlet capture rate [16,20]. As to the overlap reduction factor \( C \), the value used is that detailed in Appendix A in the case of the proton, namely \( C = 0.9956 \).

Let us turn to the issue of the nucleon electroweak form factors, to be evaluated at the momentum transfer \( q^2 = -0.877 m^2_\mu \), beginning with the vector ones, \( g_V(q^2) \) and \( g_M(q^2) \). Since through CVC, the values for these nuclear form factors are related to the electromagnetic ones for the proton and the neutron whose charge radii are very well known [45], it becomes possible to infer very precise values for \( g_V(q^2) \) and \( g_M(q^2) \). From the values \( r_1^p = 0.765(1 \pm 0.01) \) fm and \( r_1^n = 0.893(1 \pm 0.01) \) fm (see Ref. [45] for the meaning of these parameters), as well as the proton and neutron anomalous magnetic moments and the fact that \( g_V(q^2 = 0) = 1 \), the following values apply:

\[
\begin{align*}
    g_V(q^2_0) &= 0.9755 \pm 0.0005, \\
    g_M(q^2_0) &= 3.5821 \pm 0.0025.
\end{align*}
\] (76)

The value for the axial form factor \( g_A(q^2) \) is inferred from the neutron decay rate, which implies [35]

\[
    g_A(q^2 = 0) = 1.2670 \pm 0.0035,
\] (77)

as well as the mean square axial charge radius [8,10,11,46].
Consequently, one finds
\[ g_A(q_0^2) = 1.245 \pm 0.004. \tag{79} \]
Note that since the capture rates are rather sensitive to the axial form factor (see below), if the value for \( g_A(q_0^2) \) were to change again, the values for capture rates would have to be adapted appropriately.

The value for the induced pseudoscalar form factor \( g_P(q_0^2) \) is given by the prediction based on the chiral symmetries of QCD \[8–11\],
\[ g_P(q_0^2) = \frac{2m_\mu f_\pi g_{\pi NN}}{m_A^2 - q_0^2} - \frac{1}{3} g_A(0) m_\mu Mr_A^2, \tag{80} \]
where \( f_\pi = 92.5 \pm 0.2 \text{ MeV} \) is the usual pion decay constant and \( g_{\pi NN} \) the pion–nucleon coupling constant. The latter quantity has been the issue of much debate, but using the recent precise estimate of Ref. \[47\],
\[ g_{\pi NN} = 13.37 \pm 0.09, \tag{81} \]
one finds
\[ g_P(q_0^2) = 8.475 \pm 0.076, \tag{82} \]
namely, a chiral symmetry prediction with a precision better than 1%.

The second-class form factors \( g_S \) and \( g_T \) are taken to be identically vanishing, an approximation discussed in Section 2.2 which is justified in the limit of exact isospin symmetry. However, the effects of isospin breaking are estimated to be small \[38–41\], on the order of 0.02 for \( |g_S/g_V| \) and \( |g_T/g_A| \), and given the sensitivity of the capture rates to these two form factors (see below), their contribution may safely be neglected.

Finally, there is the issue of the nucleon scalar, pseudoscalar and tensor form factors \( G_S(q_0^2), G_P(q_0^2) \) and \( G_T(q_0^2) \), whose values are unknown at present. Any estimate would of course be welcome, but the expected result should be of the order of unity, within may be a factor of at most ten, as mentioned previously.

4.2. Within the standard model

Given the different inputs discussed in Section 4.1, it is straightforward to consider predictions for the observables in muon capture on the proton within the SM. Setting all effective coupling coefficients \( h^{S,V,T}_{\pm\pm} \) to zero except for \( h^{V}_{-+} = 1 \), one then finds:
\[
\begin{align*}
\lambda^S &= 688.4 \pm 3.8 \text{ s}^{-1}, & A_\Delta &= -0.9337 \pm 0.0007, \\
\lambda^T &= 12.01 \pm 0.12 \text{ s}^{-1}, & A_e &= 0.00371 \pm 0.00030, \\
\lambda^{\text{stat}} &= 181.11 \pm 0.98 \text{ s}^{-1}, & A_t &= -0.06260 \pm 0.00052.
\end{align*}
\tag{83}
\]
Note the small values for the latter two observables, rendering their experimental determination essentially impossible, given the small polarizations available for the muonic hydrogen atom.
Table 3
Sensitivities of the different observables $O = \lambda, T, \text{stat}, A_\lambda, A_v, A_t$ for muon capture on the proton, with respect to the nuclear form factors associated to the vector and axial quark operators. See (66) for the definition of $\sigma(O; F_X)$

<table>
<thead>
<tr>
<th></th>
<th>$\sigma(\lambda^S; g_X)$</th>
<th>$\sigma(\lambda^T; g_X)$</th>
<th>$\sigma(\lambda^{\text{stat}}; g_X)$</th>
<th>$\sigma(A_\lambda; g_X)$</th>
<th>$\sigma(A_v; g_X)$</th>
<th>$\sigma(A_t; g_X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_V$</td>
<td>+0.466</td>
<td>-1.129</td>
<td>+0.386</td>
<td>+0.108</td>
<td>+13.16</td>
<td>-2.39</td>
</tr>
<tr>
<td>$g_M$</td>
<td>+0.151</td>
<td>+0.680</td>
<td>+0.177</td>
<td>-0.0357</td>
<td>-0.317</td>
<td>+0.551</td>
</tr>
<tr>
<td>$g_A$</td>
<td>+1.567</td>
<td>+1.440</td>
<td>+1.561</td>
<td>+0.00856</td>
<td>-18.85</td>
<td>+0.991</td>
</tr>
<tr>
<td>$g_P$</td>
<td>-0.184</td>
<td>+1.008</td>
<td>-0.125</td>
<td>-0.0804</td>
<td>+6.01</td>
<td>+0.844</td>
</tr>
<tr>
<td>$g_S$</td>
<td>+0.0232</td>
<td>-0.0718</td>
<td>+0.0185</td>
<td>-0.00641</td>
<td>+0.724</td>
<td>-0.139</td>
</tr>
<tr>
<td>$g_T$</td>
<td>+0.0238</td>
<td>-0.125</td>
<td>+0.0164</td>
<td>+0.01003</td>
<td>-0.759</td>
<td>-0.105</td>
</tr>
</tbody>
</table>

As in the case of muon capture on $^3\text{He}$, it is interesting to consider the sensitivity of each of these observables to the different input nucleon form factors, in particular $g_P(q_0^2)$. These sensitivities are defined as in (66), and their values are presented in Table 3. Note that with respect to the measured capture rates, namely the statistical one for $^3\text{He}$ and the singlet one for the proton, their sensitivity to the induced pseudoscalar form factor is essentially comparable, as is also the case for the other nucleon form factors.

In fact, assuming the goal reached [16,20] of measuring the singlet rate on the proton to a precision of 0.5% with a central value equal to the above theoretical prediction precise to 0.55%, one may infer the following value for the induced pseudoscalar form factor:

$$g_P(q_0^2) = g_P^T(q_0^2) \pm 0.327 [\text{exp. 0.230}],$$  \hspace{1cm} (84)

where the second indicated uncertainty follows only from the experimental precision of 0.5%, while the first also includes all the errors on the theoretical inputs for the other form factors and the CKM matrix element $V_{ud}$. In other words, a 0.5% precise measurement of the singlet capture rate implies a 3.9% precise determination of $g_P$ and for the test of the chiral symmetry prediction of that value. In the same way as in Ref. [37], the same result may also be used to infer a value for the pion–nucleon coupling constant $g_{\pi NN}$, assuming that $g_P(q_0^2)$ takes its expected value (82),

$$g_{\pi NN} = 13.37 \pm 0.49,$$  \hspace{1cm} (85)

hence a 3.7% precise determination of that quantity.

Pursuing along the same lines, assuming the value for $g_P(q_0^2)$ set by the chiral symmetry prediction, and a measurement of the singlet rate precise to 0.5% in the manner described above, each of the second-class form factors may also be constrained to the following precision:

$$g_S: \pm 0.314 [\text{exp. 0.216}], \quad g_T: \pm 0.307 [\text{exp. 0.210}],$$  \hspace{1cm} (86)

thus providing much of an improvement on the present situation [25], but still an order of magnitude away from the expected range of values.
4.3. Beyond the standard model

The sensitivities of the considered observables to the effective coupling coefficients $h_{S,V,T}^{S,V,T}$ are given in Table 4, which displays some interesting differences with the $^3$He case, but again not between the singlet capture rate on the proton and the statistical rate for $^3$He.

In order to assess the potential reach offered by the singlet rate for physics beyond the SM, let us again assume a 0.5% precise measurement of that observable centered onto its theoretical prediction. One then finds the following uncertainties, for each of the relevant effective coupling coefficients $h_{S,V,T}^{S,V,T}$, turning them on each one after the other:

\[
\begin{align*}
    h_{S}^S : & \pm 0.0168 \text{[exp: 0.0115]}, \\
    h_{S}^V : & \pm 0.0187 \text{[exp: 0.0128]}, \\
    h_{S}^T : & \pm 0.0177 \text{[exp: 0.0121]}, \\
    h_{P}^S : & \pm 0.0336 \text{[exp: 0.230]}, \\
    h_{P}^V : & \pm 0.00140 \text{[exp: 0.00096]}, \\
    h_{P}^T : & \pm 0.00364 \text{[exp: 0.00250]}, \\
    h_{V}^S : & \pm 0.0113 \text{[exp: 0.0065]}, \\
    \end{align*}
\]

in which it is implicitly assumed that the form factors $G_{S,P,T}$ have all three been set to a unit value. Consequently, given the results in (73), the physics reach offered by the singlet muon capture rate on the proton measured to 0.5% precision improves by factors of at least two to three the results achieved already from the available statistical capture rate on $^3$He.

Table 4
Sensitivities of the different observables $O = \lambda^{S,T,\text{stat.}}$, $A_{\lambda}$, $A_{\nu}$, $A_{T}$ for muon capture on the proton, with respect to the effective coupling coefficients $h_{S,V,T}^{S,V,T}$. Sensitivities for those couplings which do not appear in this table are identically vanishing. The scalar and pseudoscalar combinations are defined by $h_{S}^S = h_{S}^{++} + h_{S}^{+-}$ and $h_{P}^S = h_{S}^{--} - h_{S}^{+-}$, while it is understood that all form factors $G_{S}, G_{P}$ and $G_{T}$ are set to unity to obtain the numbers in this table. See (66) for the definition of $\sigma (O; h_X)$

<table>
<thead>
<tr>
<th>$h_{\pm \pm}^S$</th>
<th>$\sigma (\lambda^S; h_X)$</th>
<th>$\sigma (\lambda^T; h_X)$</th>
<th>$\sigma (\lambda^{\text{stat}}; h_X)$</th>
<th>$\sigma (A_{\lambda}; h_X)$</th>
<th>$\sigma (A_{\nu}; h_X)$</th>
<th>$\sigma (A_{T}; h_X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>+0.434</td>
<td>-1.394</td>
<td>+0.343</td>
<td>+0.123</td>
<td>+12.16</td>
<td>-2.562</td>
<td></td>
</tr>
<tr>
<td>+0.391</td>
<td>-1.157</td>
<td>+0.314</td>
<td>+0.104</td>
<td>+13.57</td>
<td>-2.363</td>
<td></td>
</tr>
<tr>
<td>+0.412</td>
<td>-1.274</td>
<td>+0.328</td>
<td>+0.114</td>
<td>+12.86</td>
<td>-2.461</td>
<td></td>
</tr>
<tr>
<td>+0.0217</td>
<td>-0.118</td>
<td>+0.0147</td>
<td>+0.00945</td>
<td>-0.710</td>
<td>-0.0986</td>
<td></td>
</tr>
<tr>
<td>-5.21</td>
<td>-5.58</td>
<td>-5.23</td>
<td>+0.0251</td>
<td>+55.21</td>
<td>-3.65</td>
<td></td>
</tr>
<tr>
<td>+2.00</td>
<td>+2.00</td>
<td>+2.00</td>
<td>+0.00</td>
<td>+0.00</td>
<td>+0.00</td>
<td></td>
</tr>
<tr>
<td>-0.767</td>
<td>-2.90</td>
<td>-0.872</td>
<td>+0.144</td>
<td>+25.68</td>
<td>-3.669</td>
<td></td>
</tr>
</tbody>
</table>
5. Specific models beyond the standard model

The effective interaction (3) in terms of the coupling coefficients \( h_{S,V,T}^{S,V,T} \) provides a model independent parameterization for any low-energy contributions stemming from whatever new physics there is lurking behind the confines of the standard model. However, taken as such, the constraints (73) obtained from the experimental statistical capture on \( ^3\text{He} \) probably do not mean much in terms of the energy scales or coupling strengths implied by such constraints. To develop an idea for the latter kind of data however, it becomes necessary to consider specific models for physics beyond the SM. Three general such classes will briefly be considered here, with the notations defined in Appendix C. Other possibilities come to mind, such as for example models with large extra dimensions which may be amusing to investigate in the same vain. Also, when the singlet capture rate on the proton will have been measured, similar considerations may be developed as well, improving to some extent on the present results.

5.1. Left–right symmetric models

Turning first to left–right symmetric models (LRSM), only the limits on the couplings \( h_{V}^{\pm} \) and \( h_{V}^{C} \) need to be considered, since only vector and axial interactions are implied in such models (up to small scalar Higgs exchange interactions which may be ignored, see Appendix C). Given the value from (73),

\[
 h_{V}^{\pm} = 1.00090 \pm 0.00712,
\]

(88)
different considerations may be developed. First, one may view this result as a constraint on the universality of the electroweak interactions [1–3], whose most stringent limit in the electron–muon flavour sector follows from \( \pi^{\pm} \) decays [35,48] at the 0.4% level. The above limit implies a universality constraint at the 1.4% level only (in terms of \( h_{V}^{\pm} \)).

The above result for \( h_{V}^{\pm} \) may also be viewed as a constraint on the unitarity properties of the CKM quark flavour-mixing matrix, in terms of the ratio \( V_{ud}/V_{ud}^{\text{unitary}} \). Indeed, in muon capture one is dealing entirely with the muon flavour sector only, while \( V_{ud} \) usually involves different \( \beta \)-decay processes, hence the electron flavour sector. Therefore, another way to read the constraint on \( h_{V}^{\pm} \) is to say that the experimental statistical capture rate on \( ^3\text{He} \) confirms the unitary-constrained value for \( V_{ud} \), to a level better than what is achieved in terms of the \( 0^+ - 0^+ \) superallowed \( \beta \)-decays where conclusions are somewhat dependent on the nuclear models used to evaluate the radiative corrections [35].

Finally, within LRSM, the above result for \( h_{V}^{\pm} \) does not imply any limit on the mass \( M_{W}^{2} \) of the extra charged gauge boson, since an expression similar to that given in Appendix C for the coefficient \( h_{V}^{\pm} \) also applies to the similar coefficient \( f_{V}^{\pm} \) relevant to \( \beta \)-decay, so that when reexpressing all couplings in terms of physical quantities (such as the muon decay rate, and so on), such factors essentially cancel. As a matter of fact, the sole contribution which survives this comparison of muon decay, \( \beta \)-decay and muon capture amplitudes, is that in LRSM the CKM leptonic flavour-mixing matrices may be different for the electron and muon flavours. Thus in fact, the above limit on \( h_{V}^{\pm} \) translates into
the following constraint on leptonic flavour-mixing in these two sectors, when the mixing angle $\zeta$ of the charged gauge bosons is ignored, $\zeta = 0$ [17]:

$$ (v_{\mu} - v_{e}) v_{\mu} r^A \delta^2 = -0.00018 \pm 0.01424, \quad \text{with} \quad (89) $$

$$ v_{\mu} = \frac{\sum \left| U_{\mu i}^R \right|^2}{\sum \left| U_{\mu i}^L \right|^2}, \quad v_{e} = \frac{\sum \left| U_{ei}^R \right|^2}{\sum \left| U_{ei}^L \right|^2}, \quad v_{u} = \frac{|V_{ud}^R|^2}{|V_{ud}^L|^2}, \quad (90) $$
in which the summation over the index $i$ stands for all neutrino mass eigenstates whose production is kinematically allowed in muon capture on the one hand, and in $\beta$-decay on the other (in which case, their mass is taken to be negligible as well). The other parameters are defined in Appendix C.

Similarly, the result on $h^V_{\ell}$ in (73) may be translated in terms of parameters of LRSM as (see Appendix C):

$$ r t \text{Re} \left( e^{i\omega} v_{ud} \right) = 0.000222 \pm 0.0176. \quad (91) $$

In the case of manifest LRSM with $r = 1$, $v_{ud} = 1$ and $\omega = 0$, this result provides a limit on the mixing angle $\zeta$ which does not improve such limits stemming already from $\beta$-decay processes [35].

In fact, again in the limit of a vanishing mixing angle $\zeta$, the vector analyzing power $A_v$ reduces to

$$ A^\text{LRSM}_v |_{\zeta=0} = \frac{1 - r^4 \delta^2 v_{\mu} v_{u}}{1 + r^4 \delta^2 v_{\mu} v_{u}} A^\text{SM}_v, \quad (92) $$

with $A^\text{SM}_v$ its SM value. However, even a measurement to 1% of $A_v$ would not imply a lower bound on $M^W_2$ better than 260 GeV (95% C.L.) in the manifest LRSM.

5.2. Contact interactions

Let us now consider the possibility of contact interactions (see Appendix C). Given the parameterization of such interactions, it is a simple matter to translate the limits (73) in terms of the associated compositeness scales, with the following lower bounds (in an obvious notation) all given at the 95% C.L.:

$$ A^S_{+} > 1.60 \text{ TeV}, \quad A^S_{-} > 1.62 \text{ TeV}, \quad A^S_{+} > 1.61 \text{ TeV}, $$
$$ A^P_{+} > 196 \text{ GeV}, \quad A^P_{-} > 6.34 \text{ TeV}, \quad A^V_{+} > 2.36 \text{ TeV}. \quad (93) $$

Clearly, the limits on $A^S_{+}$ and $A^P_{+}$ are quite competitive with recent collider results in some of these channels [49]. One also has to keep in mind that the latter results apply to the electron sector, while those established here on basis of the experimental statistical capture rate on $^3$He apply to the muon sector for couplings between the first quark generation and the second lepton one. On the other hand, the above lower bounds are not as stringent as those which follow from atomic parity violation [50], but again the latter limits apply to the electron sector and for neutral current interactions. It is thus fair to say that precision studies in nuclear muon capture on simple nuclei have at present the potential to test the
5.3. Leptoquarks

Finally, let us consider the possibility of leptoquarks (LQ) [51,52], in the notations introduced in Appendix C. Among all the possible limits which may be obtained from (73), only the most stringent ones are presented here.

In the case of scalar LQ, the limit follows from the value for the tensor coupling coefficient $h_{T_{-,-}}^C$, leading to
\[ M > 894 \text{ GeV} \quad (95\% \text{ C.L.}). \tag{94} \]

The notation used here is symbolic. Indeed, referring back to the expressions for the $h_{S,V,T}^{\pm \pm}$ coefficients in terms of the LQ couplings and parameters, one sees that the $h_{T_{-,-}}^C$ coupling involves either the $S_0(1/3)$ LQ with the couplings $\lambda_{S_0}^L \lambda_{S_0}^R$, or the $S_{1/2}(-2/3)$ LQ with the couplings $\lambda_{S_{1/2}}^L \lambda_{S_{1/2}}^R$. Thus in the above lower bound, $M$ stands for the mass of one or the other of these scalar LQ, while $\lambda$ stands for the square root of the product of the two associated coupling constants.

Similarly considering the limit in (73) on the coefficient $h_{++}^S$, one finds the lower bound,
\[ M > 915 \text{ GeV} \quad (95\% \text{ C.L.}), \tag{95} \]
in a similar notation referring now to either vector LQ $V_0(-2/3)$ or $V_{1/2}(-1/3)$ (for the associated combination of LQ couplings corresponding to the factor $\lambda$, see Appendix C). Again, these limits are certainly as stringent as those recently presented in Refs. [49,53]. This is particularly relevant when one recalls again that the constraints from Ref. [49] refer to the electron sector, while the limits established here apply to LQ coupling the first quark generation to the second lepton generation.

6. Conclusions

This paper provides for the first time explicit and complete analytic expressions for all observables relevant to nuclear muon capture on a spin 1/2 isospin doublet, thus of direct use to the cases of the proton and $^3$He. The results include all possible contributions for the nuclear matrix elements, as well as all possible effective interactions beyond the usual electroweak charged interaction. The analysis was developed with great care, keeping approximations to the strictest minimum and in ways not affecting the final numerical results, including the calculation of nuclear finite size corrections to the muon overlap correction factor, and considering the limit of a Dirac neutrino of zero mass. Such expressions are ideally suited for tests of the standard model from precision muon capture experiments, both in addressing still open questions related to the strong nonperturbative
sector of the quark interactions probed through the electroweak sector, as well as in probing for any new physics which is lying in store just waiting to be discovered experimentally.

These results also enable specific predictions for observables with a precision on a par with the challenge on the experimental side as well. Specific new results for muon capture on the proton have been given, and some others in the case of $^3$He. This situation is particularly relevant given the recent precision measurement of the statistical capture rate on $^3$He [14], and the projected 0.5% precise measurement of the singlet rate on the proton [16,20].

The discussion showed how stringent limits can be inferred from the $^3$He result already, both within the SM and beyond it, sometimes in real competition with results from collider experiments, and often complementary to these. The physics reach of the foreseen experiment on the proton has also been assessed. In due time, when that experiment will have been completed to the desired level of precision, similar but improved limits and tests on the SM will be inferred, and in particular the issue of the induced pseudoscalar nucleon form factor settled once and for all, presumably in perfect agreement with the theoretical expectation. Also, the potential physics reach of other observables may now be assessed completely on the basis of the expressions of this paper, even though the actual measurement of any of these observables, which all involve either initial polarization muonic atom states or final polarization measurements, or both, poses an almost impossible experimental challenge. But such a situation has never deterred any experimentalist at heart, quite to the contrary! We hope that this paper will also be of use in the experimental pursuit of the impossible polarization observables.

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Appendix A. The overlap reduction factor

In the calculation as outlined in Section 2, it is implicitly assumed that one is dealing with point-like particles whose quantum states are described by plane waves. Since this is clearly not the case, further corrections have to be introduced in order to account for the finite spatial extent of the nuclear charge distributions, for the bound state character of the initial muonic atom, and for possible relativistic corrections since it is the non
relativistic Coulomb wave function $\psi_c(r)$ which is initially considered to represent the muon probability amplitude rather than the full-fledged solution to the relevant Dirac equation. Indeed, all these effects ought to be carefully assessed in order to obtain a trustworthy evaluation of the reduction factor $C$ introduced in (2), with a precision at the required level with respect to experimental aims.

This appendix presents such an evaluation, following closely the discussion developed in Ref. [4] in the case of $^3$He. Since such an analysis, which is important for our purposes, is not available in the literature, we felt it useful to include it here, the more so since it seems to have been the intention of the author of Ref. [4] to make it available.

Nuclear charge distributions, whether of electric or electroweak matter, need to be modeled in such an analysis. This we shall do in terms of a spherically symmetric density $\rho_m(r)$, where the index distinguishes the different types of matter encountered in the problem, normalized to unity over the volume of space,

$$\int_{(\infty)} d^3 r \rho_m(r) = 1,$$

and dependent on a single length scale parameter $a_m$ directly related to the mean square radius of the distribution. The simplest such model which we shall use is of the form

$$\rho_m(r) = \frac{1}{8\pi a_m^3} e^{-r/a_m},$$

such that

$$r_m^2 \equiv \langle r^2 \rangle_m = \int_{(\infty)} d^3 r r^2 \rho_m(r) = 12 a_m^2.$$  

Other charge distribution models may be considered of course [4], leading to no significant difference in the evaluation of the reduction factor $C$. Note that the form factor in momentum space associated to the model (97) is of the usual dipolar form:

$$F_m(q^2) = \frac{1}{(1 - a_m^2 q^2)^2}. \quad (99)$$

Given such models, the reduction factor $C_m$ associated to each of these nuclear matter distributions is thus given by [4]

$$\sqrt{C_m} = \int_{(\infty)} d^3 r \left( 1 + v^2 a_m^2 \right)^2 \rho_m(r) \frac{j_0(vr)}{\psi_1^{(0)}(0)}, \quad (100)$$

where $v$ stands for the neutrino energy. In this expression, the factor $(1 + v^2 a_m^2)^2$ stems from the normalization of the form factor $F_m(q^2)$ or the normalization condition for the distribution $\rho_m(r)$, $j_0(vr) = \sin(vr)/(vr)$ is the spherical Bessel function associated to the angle-integrated neutrino plane wave function $e^{-i\vec{p}\cdot\vec{r}}$, $\psi_1(r)$ is the ground state wave function of the muonic atom, and finally $\psi_1^{(0)}(0) = \psi_c(r)$ is the 1S ground state Coulomb wave function of the muonic atom since this specific choice was made in the normalization
of the capture distribution (2). Hence, only the function \( \psi_1(r) \) is left to be computed in order to evaluate the above overlap integral giving the reduction factor \( C_m \).

The wave function \( \psi_1(r) \) is that of the muon ("carrying" the reduced mass \( \mu \) of the muonic atom) in the electrostatic field of the nucleus of finite size. In principle, this function is to be obtained by solving the associated Dirac equation in the given electrostatic potential, but one would expect that the ensuing relativistic corrections should be sufficiently small to be ignored since of order \((\alpha Z)^2\), which is the squared velocity of the bound muon. This expectation is indeed borne out by the detailed numerical resolution of the associated Dirac equation in the case of \(^3\text{He}\) [4] and of the proton, and this relativistic correction shall thus not be included here in the evaluation of \( C_m \).

Hence, this leaves only to solve the Schrödinger equation in the same electrostatic potential of the electric charge distribution of the initial nucleus. For the latter distribution, we shall again use a model of the form (97), with a parameter \( a_c \) related to the electric charge mean square radius of that nucleus. Correspondingly, in addition to the usual Coulomb potential for a point charge of value \( +Ze \), one needs to include in the Schrödinger equation the following perturbation in the potential [4]:

\[
\Delta V(r) = aZhc \left( \frac{1}{r} + \frac{1}{2a_c} \right) e^{-r/a_c}.
\]

The contributions of this term to the 1S ground state wave function \( \psi_1(r) \) are then computable through perturbation theory. It appears that the first order correction is already sufficient for our purposes since it is proportional to the factor \((a_c/a_0)^2 \sim 10^{-5}\), where \( a_0 \) is the usual atomic Bohr radius for the muonic atom, thus given by

\[
a_0 = \frac{\hbar c}{\alpha Z \mu c^2}.
\]

The spherically symmetric solutions to the Coulomb problem are well known. For the \( nS \) state, \( n = 1, 2, \ldots \), one has

\[
\psi_{n}^{(0)}(r) = \frac{1}{\sqrt{\pi a_0^3 n^2}} e^{-r/(na_0)} L_{n-1}^1 \left( \frac{2r}{na_0} \right), \quad \psi_c(0) \equiv \psi_{1}^{(0)}(0) = \frac{1}{\sqrt{\pi a_0^3}},
\]

where \( L_{n-1}^1(x) \) are the usual Laguerre polynomials. The associated (binding) energy eigenvalues are

\[
E_n^{(0)} = \frac{1}{2} (\alpha Z)^2 \mu c^2 \frac{1}{n^2}.
\]

Adding the correction (101), first-order perturbation theory then leads to the following 1S ground state normalized wave function:

\[
\psi_1(r) = \psi_1^{(0)}(r) - 16 \left( \frac{a_c}{a_0} \right)^2 \sum_{n=2}^{\infty} \frac{1}{\sqrt{n^2}} \frac{1}{n^2} \psi_{n}^{(0)}(r),
\]

while the associated energy is

\[
E_1 = -\frac{1}{2} (\alpha Z)^2 \mu c^2 \left( 1 - 16 \left( \frac{a_c}{a_0} \right)^2 \right).
\]
The evaluation of the overlap reduction factor $C_m$ in (100) is now straightforward using the integrals

$$\frac{1}{\psi_{1}^{(0)}(0)} \int_{(\infty)}^{} \rho_{m}(r) \psi_{n}^{(0)}(r) = \frac{1}{\sqrt{\pi}} \left(1 + \frac{1}{n} \frac{a_{m}}{a_{0}}\right)^{n-2} \left(1 - \frac{a_{m}}{a_{0}}\right).$$

(107)

One then finds

$$\sqrt{C_{m}} = \frac{1}{1 + \frac{a_{m}}{a_{0}}} \left(1 + \frac{1 + v^{2}a_{m}^{2}}{1 + v^{2}a_{0}^{2}}\right)^{2} - 4 \left(\frac{a_{c}}{a_{0}}\right)^{2},$$

(108)

where in the second term which follows from the first-order correction in (105) the limit $a_{m} \to 0$ has been taken. Numerically, this result does not differ from the same expression expanded to second order\textsuperscript{12} in $a_{c}/a_{0}$ and $a_{m}/a_{0}$ [4]:

$$\sqrt{C_{m}} = 1 - 3 \frac{1 - v^{2}a_{m}^{2}}{1 + v^{2}a_{m}^{2}} \left(\frac{a_{m}}{a_{0}}\right) + 6 \frac{1 - v^{2}a_{m}^{2}}{1 + v^{2}a_{m}^{2}} \left(\frac{a_{m}}{a_{0}}\right)^{2} - 4 \left(\frac{a_{c}}{a_{0}}\right)^{2}.$$

(109)

Let us now turn to numerical evaluations, first in the case of $^{3}$He. The values for the relevant parameters $a_{m}$ may only be inferred from a nuclear model calculation of the corresponding mean square charge radii. This is done in Ref. [4] with the following results:

$$a_{c(1)} = 0.554 \text{ fm}, \quad a_{c(\sigma)} = 0.512 \text{ fm},$$

(110)

where $a_{c(1)}$ (respectively, $a_{c(\sigma)}$) stands for the parameter $a_{m}$ associated to form factors for the vector (respectively, axial) current (thus including the electromagnetic current in the vector case). Using then the mass values and neutrino energy given in Section 3.1, one finds:

$$\frac{a_{c}}{a_{0}} = 4.17 \cdot 10^{-3}, \quad C_{c(1)} = 0.9777, \quad C_{c(\sigma)} = 0.9790.$$  

(111)

Weighing these two reduction factors with the relative vector and axial current contributions to the capture rates, an effective value of $C_{\text{eff}} = 0.9788$ is obtained, thus finally leading to the overlap reduction factor for $^{3}$He as determined in Ref. [4],

$$C^{(3)\text{He}} = 0.979.$$  

(112)

In the case of the proton, things may be done with great precision since a great deal is known about the nucleon electromagnetic form factors [8,45,46]. Given the value $r_{E}^{p} = 0.847 \text{ fm}$ [45] (known to 1% precision) for the proton electric charge distribution, one has

$$\frac{a_{c}}{a_{0}} = 8.59 \cdot 10^{-4}.$$  

(113)

On the other hand, the values for $C_{V,M,A}$ may be determined using the associated charge radii $r_{V}^{1}$, $r_{V}^{2}$ and $r_{A}$ discussed in Section 4.1, including their uncertainties. One then finds:

\textsuperscript{12} Both these ratios being on the order of $10^{-3}$, cubic corrections are indeed totally negligible for our purposes.
Assuming then that the correction factor associated to the nuclear matter distribution characterized by the induced pseudoscalar form factor $g_P$ is also given by $C_A$, and weighing each of these correction factors with the respective relative contributions of the associated form factors to the capture rates, the following effective value for the overlap reduction factor is finally obtained:

$$C(p) = 0.9956.$$  \hspace{1cm} (115)

Note that this value agrees essentially with the 0.4% correction quoted in Ref. [23] in the case of hydrogen. Here however, a careful assessment of all possible effects has led to this final value, with increased precision.

**Appendix B. Polarization and hyperfine states**

In the calculation outlined in Section 2, the natural representation of polarization states is through the normalized spin vectors $\hat{s}_{\mu,1,2}$ rather than the hyperfine states $(S = 1, m = 0, \pm 1)$ and $(S = 0, m = 0)$ of the muonic atom. However, there exists a correspondence between these two bases for the initial spin degrees of freedom, and the purpose of this appendix is to indicate how results pertaining to these hyperfine states may be extracted from the results derived in Section 2.

First, let us note that for a spin $1/2$ system, any of its normalized quantum states may be parameterized as

$$|\hat{s}\rangle = e^{-i\varphi/2} \cos \frac{\theta}{2} |+\rangle + e^{i\varphi/2} \sin \frac{\theta}{2} |-\rangle,$$  \hspace{1cm} (116)

where $|\pm\rangle$ represent the basis of states with spin eigenvalues $m = \pm 1/2$ with respect to an arbitrary quantization axis. The angular variables $(\theta, \varphi)$ may be interpreted as being the spherical coordinates for a unit vector $\hat{s}$ whose components are defined as in (15), since one finds for the associated spin component operators given in terms of the usual Pauli matrices $\sigma$:

$$\langle \vec{\sigma} \rangle \equiv \langle \hat{s}| \vec{\sigma} |\hat{s}\rangle = \hat{s}$$ \hspace{1cm} (117)

(note also that the state $|\hat{s}\rangle$ corresponds to the bi-spinor $\chi_{\mu}(\hat{s})$ in (16)).

Let us now consider the quantum spin states of a bound system of two spin $1/2$ particles, such as the muonic atom, obtained through the tensor product $|\hat{s}_{\mu}\rangle_{\mu}|\hat{s}_1\rangle_{1}$ of the associated spin states:

$$|\hat{s}_{\mu}\rangle_{\mu} = e^{-i\varphi_{\mu}/2} \cos \frac{\theta_{\mu}}{2} |+\rangle_{\mu} + e^{i\varphi_{\mu}/2} \sin \frac{\theta_{\mu}}{2} |-\rangle_{\mu},$$  \hspace{1cm} (118)

$$|\hat{s}_1\rangle_1 = e^{-i\varphi_1/2} \cos \frac{\theta_1}{2} |+\rangle_1 + e^{i\varphi_1/2} \sin \frac{\theta_1}{2} |-\rangle_1.$$  \hspace{1cm} (119)

In terms of the usual $|S, m\rangle$ hyperfine state,
hyperfine capture amplitudes expressed in Section 2.2 (in a notation which should be self-explanatory). In terms of

The hyperfine capture distributions are then given as in (2) with the quantity $j$

\[ N \]

with $c$

\[ jO \]

more involved however, because of their intertwined character in terms of the spherical coordinates $s$

one then finds:

\[ j(\mu)_{\mu}|j(\mu)| = A_{1,1}|1, 1) + A_{1,-1}|1, -1) + A_{1,0}|1, 0) + A_{0,0}|0, 0), \quad \text{where} \]

(122)

\[ A_{1,1} = e^{-i((\theta)_{\mu} + r_{\mu})/2}c_{\mu}, \]

(123)

\[ A_{1,-1} = e^{i(r_{\mu} - (\theta)_{\mu})/2}s_{\mu}, \]

(124)

\[ A_{1,0} = e^{i(r_{\mu} - (\theta)_{\mu})/2}c_{\mu}s_{1}, \]

(125)

\[ A_{0,0} = e^{i(r_{\mu} - (\theta)_{\mu})/2}s_{\mu}c_{1}. \]

with $c_{\mu} = \cos \theta_{\mu}/2$, $s_{\mu} = \sin \theta_{\mu}/2$, $c_{1} = \cos \theta_{1}/2$ and $s_{1} = \sin \theta_{1}/2$. In particular, the hyperfine populations are then given as

\[ N_{1,1} = \frac{1}{2}[1 + \cos \theta_{\mu}][1 + \cos \theta_{1}], \quad N_{1,-1} = \frac{1}{2}[1 - \cos \theta_{\mu}][1 - \cos \theta_{1}], \]

(126)

\[ N_{1,0} = \frac{1}{2}[1 - \cos \theta_{\mu} \cos \theta_{1} + \cos(\phi_{\mu} - \phi_{1}) \sin \theta_{\mu} \sin \theta_{1}], \]

(127)

\[ N_{0,0} = \frac{1}{2}[1 - \cos \theta_{\mu} \cos \theta_{1} - \cos(\phi_{\mu} - \phi_{1}) \sin \theta_{\mu} \sin \theta_{1}], \]

(128)

with $N_{1,1} + N_{1,-1} + N_{1,0} + N_{0,0} = 1$, as it should.

Let us now consider the matrix element $M_{\lambda}(j(\mu), j(\mu)) = (j(2, \lambda)|R_{\mu}|j(\mu), j(\mu))$ which was expressed in Section 2.2 (in a notation which should be self-explanatory). In terms of the above change of basis in spin space, one then finds the following decomposition into hyperfine capture amplitudes $M^{1,m}_{\lambda}$:

\[ M_{\lambda}(\mu, j) = A_{1,1}M^{1,1}_{\lambda} + A_{1,-1}M^{1,-1}_{\lambda} + A_{1,0}M^{1,0}_{\lambda} + A_{0,0}M^{0,0}_{\lambda}. \]

(129)

The hyperfine capture distributions are then given as in (2) with the quantity $|M_{\lambda}|^{2}$ replaced by $|M^{1,m}_{\lambda}|^{2}$. However, the former quantity evaluated in terms of the latter also involves interference contributions from different hyperfine amplitudes, which must be disposed of.

In the case of the $(S = 1, m = \pm 1)$ hyperfine states, this is straightforward, since one has

\[ |M_{\lambda}^{1,\pm 1}|^{2} = |M_{\lambda}(\pm \hat{e}_{1}, \pm \hat{e}_{2}, \hat{e}_{3})|^{2}, \]

(130)

where the right-handed orthonormalized basis $\{\hat{e}_{1}, \hat{e}_{2}, \hat{e}_{3}\}$ with respect to which the spherical coordinates $(\theta, \phi)$ are defined, has been introduced.

The situation for the two other hyperfine states $(S = 1, m = 0)$ and $(S = 0, m = 0)$ is more involved however, because of their intertwined character in terms of the $|j(\mu)_{\mu}|j(\mu)|$ spin states. One way to proceed is as follows.

Consider the following specific combinations:
\[ X = |\mathcal{M}_x(\hat{e}_3, -\hat{e}_3)|^2 + |\mathcal{M}_x(-\hat{e}_3, \hat{e}_3)|^2 = |\mathcal{M}_x^{1,0}|^2 + |\mathcal{M}_x^{0,0}|^2, \]  
\[ Y = |\mathcal{M}_y(\hat{e}_1, -\hat{e}_1)|^2 - |\mathcal{M}_y(-\hat{e}_1, \hat{e}_1)|^2 - |\mathcal{M}_z(-\hat{e}_1, -\hat{e}_1)|^2 + |\mathcal{M}_z(\hat{e}_1, \hat{e}_1)|^2 = |\mathcal{M}_z^{1,0}|^2 - |\mathcal{M}_z^{0,0}|^2 + 2 \text{Re}(\mathcal{M}_z^{1,1} \mathcal{M}_z^{-1,1}), \] 
and
\[ Z = \frac{\partial^2}{\partial \varphi_1^2} |\mathcal{M}_j(\cos \varphi_\mu \hat{e}_1 + \sin \varphi_\mu \hat{e}_2, \cos \varphi_1 \hat{e}_1 + \sin \varphi_1 \hat{e}_2)|^2 |_{\varphi_\mu = 0, \varphi_1 = 0} 
\begin{align*}
&= \frac{1}{4} \left[ |\mathcal{M}_j^{1,0}|^2 - |\mathcal{M}_j^{0,0}|^2 - 2 \text{Re}(\mathcal{M}_j^{1,1} \mathcal{M}_j^{-1,1}) \right].
\end{align*}  \] 

Then it follows obviously that the remaining two hyperfine state capture distributions are obtained from
\[ |\mathcal{M}_x^{1,0}|^2 = \frac{1}{2} [X + \frac{1}{2} (Y + 4Z)], \quad |\mathcal{M}_x^{0,0}|^2 = \frac{1}{2} [X - \frac{1}{2} (Y + 4Z)]. \] 

Applying this procedure to the quantity \( \mathcal{N}_x \) defined in (42) then leads to the hyperfine state distributions detailed in Section 2.4. Similarly, the quantity \( \mathcal{D}_x \) defined in (42) would lead to the final nucleus polarization state associated to capture from each of the muonic hyperfine states, as given in (44).

**Appendix C. Left–right symmetric models, leptoquarks and contact interactions**

The purpose of this appendix is to provide explicit expressions for the effective coupling coefficients \( h_{S,V,T}^{S,V,T} \) introduced in (3) in terms of the parameters of specific models or parameterizations for physics beyond the SM. This allows for model independent bounds that could be determined for the \( h_{S,V,T}^{S,V,T} \) coefficients from some given experiment, to be translated into may be more physically tangible numbers to be compared with the reach of other experiments, especially at high energy colliders. Three general classes of such models beyond the SM are considered here, namely, left–right symmetric models (LRSM), contact interactions and leptoquarks [51,52].

**Left–right symmetric models**

In the case of LRSM, we refer to the notations, discussion and references in Ref. [35]. For the process of interest in this paper, in the limit that Higgs exchange is ignored, which ought to be justified given the necessarily large masses for such particles in LRSM as well as their small Yukawa couplings directly proportional to the masses of the quarks and leptons involved in the process, only \( V \) and \( A \) couplings arise in addition to the purely \((V - A)\) character of the SM charged electroweak interactions. Two massive charged gauges bosons \( W_{1,2} \) appear, with masses \( M_{W_{1,2}} \), which, up to their mixing through an angle \( \alpha \) possibly accompanied by a CP violating phase \( \omega \), are associated to \((V - A)\) and \((V + A)\) interactions characterized by gauge coupling constants \( g_{L,R} \). In the fermionic sector, flavour mixing is also parameterized by Cabibbo–Kobayashi–Maskawa mixing matrices, associated to each chirality sector, denoted \( V_{L,R} \) and \( U_{L,R} \) for the quark and lepton.
sectors, respectively (indeed in a generic LRSM, neutrinos are massive and thus flavour-mix with one another as the quarks do). Let us introduce the following combinations of parameters:

$$\delta = \left(\frac{M_{W_1}^2}{M_{W_2}^2}\right)^2, \quad r = \frac{g_R}{g_L}, \quad t = \tan \xi, \quad v_{ud} = \frac{V_{ud}^R}{V_{ud}^L}, \quad \rho = \frac{g_L^2}{(M_1^2)} \cos^2 \xi \, v_{ud}^L.$$ (135)

Given the parameterization (3), one then finds for the only nonvanishing effective coupling coefficients:

$$\frac{g^2}{8M^2} \mathcal{h}_{V}^{\mu} = \rho (1 + \delta t^2) U_{\mu i}^L,$$

$$\frac{g^2}{8M^2} \mathcal{h}_{V}^{\mu} = \rho r t (1 - \delta) \epsilon^{\mu \nu} U_{\mu i}^L v_{ud},$$

$$\frac{g^2}{8M^2} \mathcal{h}_{V}^{\mu} = \rho r^2 (t^2 + \delta) U_{\mu i}^R v_{ud},$$

$$\frac{g^2}{8M^2} \mathcal{h}_{V}^{\mu} = \rho r (1 - \delta) \epsilon^{\mu \nu} U_{\mu i}^R.$$ (136)

Here, the indices $i = 1, 2, 3$ and $\mu$ on the neutrino CKM matrix elements $U_{\mu i}^{L,R}$ stand for the neutrino mass and muon flavour eigenstates, respectively.

**Contact interactions**

Let us now turn to the parameterization of so-called contact interactions, which provide a simple-minded model to probe the scale for compositeness of quarks and leptons. Such interactions are typically represented through an effective four-Fermi coupling of the form [35], say in the case of vector operators,

$$4\epsilon_{\eta_1, \eta_2} \frac{g_c^2}{8\Lambda^4} \bar{\psi}^{\eta_1} \gamma_{\mu} P_{\eta_1} \psi \bar{\psi}^{\eta_2} \gamma^{\mu} P_{\eta_2} \psi,$$ (138)

where $\epsilon_{\eta_1, \eta_2} = \pm 1$, $g_c$ is a contact interaction coupling constant, $\Lambda$ is the associated energy scale, and $P_{\eta_1, \eta_2}$ are the chirality projectors already introduced in (3). In the case of nuclear muon capture, the different spinor fields appearing in this expression correspond of course to those of (3), namely the $u$ and $d$ quarks, and the muon and its associated neutrino. It is conventional [35] to fix the scale $\Lambda$ by setting $g_c^2 = 4\pi$ (which in the case of QED would amount to having the fine structure constant set to unity, $\alpha = 1$).

Clearly, such contact interactions may be introduced for any of the scalar, vector and tensor couplings included in (3), and for any combination of the fermion chiralities involved. Thus, one could associate a scale $\Lambda$ say to vector interactions of $LL$, $VV$, $VA$, etc., chiralities, in any combination possible, and similarly for scalar and tensor interactions. Note that in the case of nuclear muon capture, these contact interactions are all related to processes which couple the quarks of the first generation to the leptons of the second.
Leptoquarks

Finally, let us consider the case of leptoquarks (LQ) [35,51,52]. These particles come in two varieties, namely either spin 0 or spin 1, and their name derives from the fact that they couple always to a quark and a lepton in a single vertex. When one allows for right-handed neutrinos as well (which was not considered in the original discussion [51]), there are six different types of scalar and of vector LQ, characterized by their weak isospin and electric charge. Their quantum numbers under $SU(3)_c \times SU(2)_L \times U(1)$ are as follows, for scalar LQ:

\[
S_0: \quad (3, 1, -2/3), \quad Q: \quad (-1/3), \\
\tilde{S}_0: \quad (3, 1, -8/3), \quad Q: \quad (-4/3), \\
S_{0v}: \quad (3, 1, 4/3), \quad Q: \quad (2/3), \\
S_{1/2}: \quad (3, 2, -7/3), \quad Q: \quad (-2/3, -5/3), \\
\tilde{S}_{1/2}: \quad (\bar{3}, 2, -1/3), \quad Q: \quad (1/3, -2/3), \\
S_1: \quad (3, -2/3), \quad Q: \quad (2/3, -1/3, -4/3),
\]

and for vector LQ:

\[
V_0: \quad (\bar{3}, 1, -4/3), \quad Q: \quad (-2/3), \\
\tilde{V}_0: \quad (3, 1, -10/3), \quad Q: \quad (-5/3), \\
V_{0v}: \quad (3, 1, 2/3), \quad Q: \quad (1/3), \\
V_{1/2}: \quad (3, 2, -5/3), \quad Q: \quad (-1/3, -4/3), \\
\tilde{V}_{1/2}: \quad (3, 2, 1/3), \quad Q: \quad (2/3, -1/3), \\
V_1: \quad (3, 3, -4/3), \quad Q: \quad (1/3, -2/3, -5/3),
\]

where each time the electric charge content of the associated isospin multiplet is given on the right. Note that the lower index carried by each of these fields labels its weak isospin value. Moreover, $\tilde{S}_{0v}$ and $V_{0v}$ are those LQ related to the introduction of right-handed neutrinos.

The scalar LQ interactions are then parameterized according to the Lagrangian density:

\[
\mathcal{L}_S = \lambda^S_{\tilde{S}_0} q_L \bar{\psi}_L \tau_2 \ell_L S_0^\dagger + \lambda^{R\tilde{R}}_{S_0} u_R \mu_R S_0^\dagger + \lambda^{R\tilde{R}}_{\tilde{S}_0} d_R \nu_R S_0^\dagger + \lambda^{R\tilde{R}}_{S_{0v}} u_R \nu_R \tilde{S}_{0v}^\dagger
+ \lambda^{R\tilde{R}}_{S_{1/2}} u_R \eta_1 \ell_L S_{1/2}^\dagger + \lambda^{R\tilde{R}}_{\tilde{S}_{1/2}} q_L \ell_L \eta_2 S_{1/2}^\dagger \tau_2 \nu_R + \lambda^{R\tilde{R}}_{S_1} u_R \eta_2 S_1^\dagger
+ \lambda^{R\tilde{R}}_{\tilde{S}_1} q_L \eta_2 S_1^\dagger \cdot \tau_2 \bar{\psi}_L + \text{h.c.} \quad (139)
\]

and similarly for vector LQ:

\[
\mathcal{L}_V = \lambda^V_{V_0} q_L \bar{\psi}_L \eta_1 \ell_L V_0^\dagger + \lambda^{R\tilde{R}}_{V_0} u_R \mu_R V_0^\dagger + \lambda^{R\tilde{R}}_{\tilde{V}_0} d_R \nu_R V_0^\dagger + \lambda^{R\tilde{R}}_{V_{0v}} u_R \nu_R \tilde{V}_{0v}^\dagger
+ \lambda^{R\tilde{R}}_{V_{1/2}} u_R \eta_1 \ell_L V_{1/2}^\dagger + \lambda^{R\tilde{R}}_{\tilde{V}_{1/2}} q_L \ell_L \eta_2 V_{1/2}^\dagger \tau_2 \nu_R + \lambda^{R\tilde{R}}_{V_1} u_R \eta_2 V_1^\dagger
+ \lambda^{R\tilde{R}}_{\tilde{V}_1} q_L \eta_2 V_1^\dagger \cdot \eta_1 \bar{\psi}_L + \text{h.c.} \quad (142)
\]
Here, $L$ and $R$ stand for definite chiral components of the spinor fields, the upper index "c" refers to the charge conjugate fields, and $q_L$ and $\ell_L$ stand for the following quark and lepton doublets:

$$q_L: \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \ell_L: \begin{pmatrix} v_L \\ \mu_L \end{pmatrix}. \quad (143)$$

Finally, the different $\lambda^{L,R}_{S,V}$ coefficients are complex constant parameters, the LQ coupling constants. In general, these coefficients are matrices in generation space, but in the case of nuclear muon capture, only those LQ couplings between the first quark generation and the second lepton generation, including the possibility of right-handed neutrinos, are relevant, hence our choice of notation.

Given these definitions, the induced nonvanishing effective couplings coefficients $h_{\pm\pm}^{S,V,T}$ in (3) are expressed as

$$\frac{g^2}{8M^2} V_{ad} (\phi_{--}^S)^* = -\frac{1}{2} \frac{\lambda^{L+}_0 \lambda_{V0}^R}{M_{V_0}^2 (-2/3)} - \frac{1}{2} \frac{\lambda^{L+}_{V_1/2} \lambda_{V1/2}^R}{M_{V_1/2}^2 (-1/3)} \quad (144)$$

$$\frac{g^2}{8M^2} V_{ad} (\phi_{--}^S)^* = -\frac{1}{8} \frac{\lambda^{L+}_{S_0} \lambda_{S_0}^R}{M_{S_0}^2 (1/3)} + \frac{1}{8} \frac{\lambda^{L+}_{S_{1/2}} \lambda_{S_{1/2}}^R}{M_{S_{1/2}}^2 (-2/3)} \quad (145)$$

$$\frac{g^2}{8M^2} V_{ad} (\phi_{--}^S)^* = -\frac{1}{8} \frac{\lambda^{L+}_{S_0} \lambda_{S_0}^R}{M_{S_0}^2 (1/3)} - \frac{1}{8} \frac{\lambda^{L+}_{S_{1/2}} \lambda_{S_{1/2}}^R}{M_{S_{1/2}}^2 (-2/3)} \quad (146)$$

$$\frac{g^2}{8M^2} V_{ad} (\phi_{--}^S)^* = -\frac{1}{2} \frac{\lambda^{L+}_{Y_0} \lambda_{Y_0}^R}{M_{Y_0}^2 (-2/3)} - \frac{1}{2} \frac{\lambda^{L+}_{Y_1/2} \lambda_{Y1/2}^R}{M_{Y_1/2}^2 (-1/3)} \quad (147)$$

$$\frac{g^2}{8M^2} V_{ad} (\phi_{--}^S)^* = -\frac{1}{8} \frac{|\lambda^{L}_{S_0}|^2}{M_{S_0}^2 (1/3)} - \frac{1}{8} \frac{|\lambda^{L}_{S_{1/2}}|^2}{M_{S_{1/2}}^2 (-1/3)} + \frac{1}{4} \frac{|\lambda^{L}_{V_0}|^2}{M_{V_0}^2 (-2/3)} \quad (148)$$

$$\frac{g^2}{8M^2} V_{ad} (\phi_{++}^S)^* = -\frac{1}{8} \frac{\lambda^{R+}_{S_0} \lambda_{S_0}^L}{M_{S_0}^2 (1/3)} + \frac{1}{4} \frac{\lambda^{R+}_{V_0} \lambda_{V_0}^L}{M_{V_0}^2 (-2/3)} \quad (149)$$

$$\frac{g^2}{8M^2} V_{ad} \frac{1}{2} (\phi_{--}^T)^* = -\frac{1}{32} \frac{\lambda^{R+}_{S_0} \lambda_{S_0}^L}{M_{S_0}^2 (1/3)} + \frac{1}{32} \frac{\lambda^{R+}_{S_{1/2}} \lambda_{S_{1/2}}^L}{M_{S_{1/2}}^2 (-2/3)} \quad (150)$$

$$\frac{g^2}{8M^2} V_{ad} \frac{1}{2} (\phi_{++}^T)^* = -\frac{1}{32} \frac{\lambda^{R+}_{S_0} \lambda_{S_0}^L}{M_{S_0}^2 (1/3)} + \frac{1}{32} \frac{\lambda^{R+}_{S_{1/2}} \lambda_{S_{1/2}}^L}{M_{S_{1/2}}^2 (-2/3)} \quad (151)$$

In these expressions, which of the LQ isospin component contributes to each coefficient is indicated by giving its electric charge in the parenthesis following the mass value in the mass contributions $1/M^2_{S,V}$. Note that the $\tau_{3b'}$ and $V_{0}$. LQ do not contribute to these coefficients.
References

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